Lagrangian Averaged Euler equations and application to Vortex Sheets

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What are the Lagrangian averaged Euler equations?

- LAE equations first appeared in the literature (Holm, Marsden, Ratiu, 1998) as n-dimensional generalization of the 1-D Camassa-Holm (CH) equations, origins in shallow water theory (Holm, 1993)
- Chen *et al* (1998) introduced phenomenological viscosity and proposed the LANS equation as a closure approximation for the RANS equations for incompressible fluid flows
- Tested by finding analytical solutions for the VCHE in turbulent channel and pipe flows and found a match at moderate to high Reynolds number for most of the flow region
- Isotropic form of LAE/LANS is (with $\nabla \cdot u = 0$)

$$\partial_t(1-\alpha^2\Delta)u+(u\cdot\nabla)(1-\alpha^2\Delta)u+[\nabla u]^T\cdot(1-\alpha^2\Delta)u = -\nabla p+\{\nu\Delta(1-\alpha^2\Delta)u\}$$

Vorticity form of the LAE

$$\partial_t q + u \cdot \nabla q = q \cdot \nabla u$$

where $q = \nabla \times v$ is the vorticity and $u = G * v$ is a filtered velocity. G is such that

$$v = (1 - \alpha^2 \Delta) u \ \text{ i.e. } \ G = \mathcal{F}^{-1}\left(\frac{1}{1 + \alpha^2 k^2}\right)$$

- Fourier space: suppression of small scales, Still some leakage, Steeper drop-off than Kolmogorov
- Small-scale vorticity is advected by the filtered velocity

Some properties of the LAE

- Circulation theorem, Conservation of energy & helicity
- The LAE equations describe geodesic motion on the volume preserving diffeomorphism group for a metric that contains the H^1 norm rather than the L^2 norm of the mean fluid velocity
- Formally same as the equations of motion for an inviscid non-Newtonian fluid of second grade - different derivation
- The transport velocity is smoothed by Helmholtz operator $(1 - \alpha^2 \Delta)$ which contains a length scale corresponding to the magnitude of the fluctuation covariance
- Steep gradients of v don't steepen further and thin vortex tubes don't get thinner as they are transported
- Effect on length scales larger than α is negligible. Consistent

Periodic vortex sheet problem

A vortex sheet may be viewed as the limiting case of a thin free shear layer as viscosity and thickness $\to 0$. Position of the curve as a function Z of the circulation Γ is

$$\frac{\partial \bar{Z}}{\partial t}(\Gamma, t) = \frac{-i}{2\pi} \int \frac{d\Gamma'}{Z(\Gamma, t) - Z(\Gamma', t)}$$

Periodic case: Birkhoff-Rott (1962,1956) system

$$\frac{\partial \bar{Z}}{\partial t}(\Gamma, t) = \frac{1}{2i} \int_0^1 \cot \pi (Z(\Gamma, t) - Z(\Gamma', t)) d\Gamma'$$

- Sulem *et al* (1981): analytic data $\Rightarrow$ analytic solution till some $t > 0$
- Moore (1979): Singularity in curvature in time $t_c < \infty$
- Pullin's conjecture: Double branched spiral for $t > t_c$, infinite turns, vanishes in size as $t \to t_c^+$

Ill-posedness of vortex sheet

$$(u,v)(x,y) = \frac{1}{2}\int_0^1 \Gamma(x',y')\frac{(-\sinh 2\pi(y-y'), \quad \sin 2\pi(x-x'))}{\cosh 2\pi(y-y') - \cos 2\pi(x-x')}$$

Linear stability analysis gives dispersion relation $\sigma(k) = k/2$, ill-posed

Numerical studies

- Rosenhead (1931): Divide circulation among finite set of vortices
- Birkhoff (1962): Results suspect, Instability from roundoff

Regularization Small parameter δ^2 in the denominator R. Krasny (1986)

$$(u,v)(i) = \frac{1}{2}\sum_{j=1, j\neq i}^{N} \Gamma(j)\frac{(-\sinh 2\pi(y_i-y_j), \sin 2\pi(x_i-x_j))}{\cosh 2\pi(y_i-y_j) - \cos 2\pi(x_i-x_j) + \delta^2}$$

Widely used with success, $\delta \to 0$ not proved, no clear interpretation for δ

LAE - Vortex blob connection

LAE is equivalent to the vortex blob method with a certain blob shape: $\zeta^\alpha(\mathbf{x}) = \frac{1}{2\pi\alpha^2}K_0(\frac{r}{\alpha})$. To see this,

$$v = (1-\alpha^2\Delta)u \;\Rightarrow u = G^\alpha * v := \mathcal{F}^{-1}(\frac{1}{1+\alpha^2 k^2}) * v$$

$$u = K * G^\alpha * q = K^\alpha * q = K * \zeta_\alpha * q$$

The system of equations now is

$$\partial_t q + u\cdot\nabla q = 0 \tag{1}$$

$$u = K^\alpha * q; \quad q(0) = q_0 \tag{2}$$

Oliver and Shkoller (2001) - Point vortices are meaningful as initial data, convergence to Euler as $\alpha \to 0$ under certain (rather weak) conditions.

$$K^\alpha = \frac{1}{2\pi r^2}(-y,x)(1-\frac{r}{\alpha}K_1(\frac{r}{\alpha})$$

LAE: discretized periodic sheet

$$(u,v)(i) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \Gamma(j) \sum_{j=1, j\neq i}^{N} \frac{(-\bar{y}, \bar{x})}{r_n^2} [1 - \frac{r_n}{\alpha} K_1(\frac{r_n}{\alpha})]$$

where $\bar{y} = y_i - y_j, \quad \bar{x} = x_i - x_j - n, \quad r_n^2 = \bar{y}^2 + \bar{x}^2$

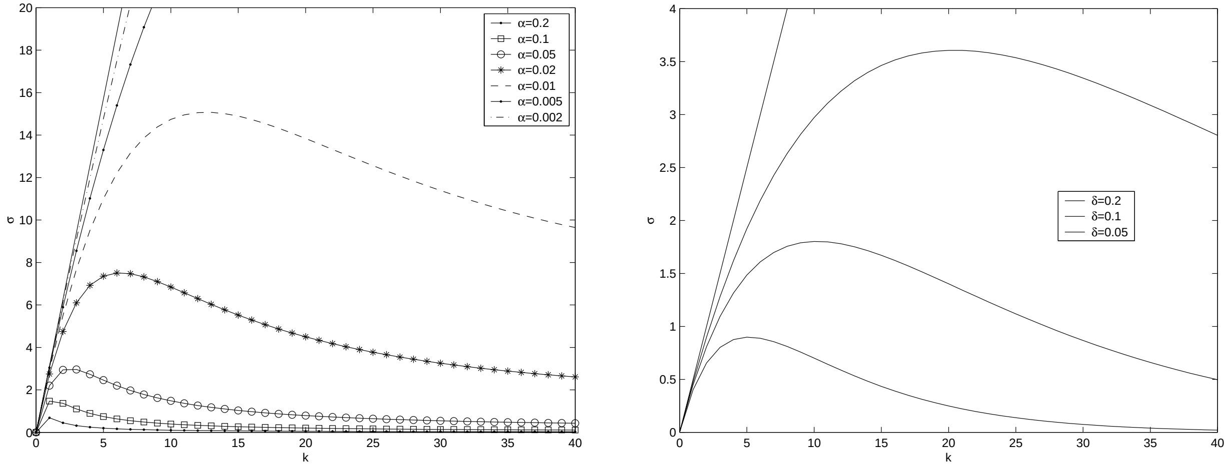
First term exact, second term truncated (the series converges)

$$(u,v)(i) = \frac{1}{2} \sum_{j=1, j\neq i}^{N} \Gamma(j) \frac{(-\sinh 2\pi\bar{y}, \sin 2\pi(x_i - x_j))}{\cosh 2\pi\bar{y} - \cos 2\pi(x_i - x_j)}$$

$$- \frac{1}{2\pi} \sum_{j=1, j\neq i}^{N} \Gamma(j) \sum_{n=-P}^{P} \frac{(-\bar{y}, \bar{x})}{r_n^2} \frac{r_n}{\alpha} K_1(\frac{r_n}{\alpha})$$

RK2, Point insertion with distance and angle criteria, local cubic polynomial interpolation, redistribution of circulation

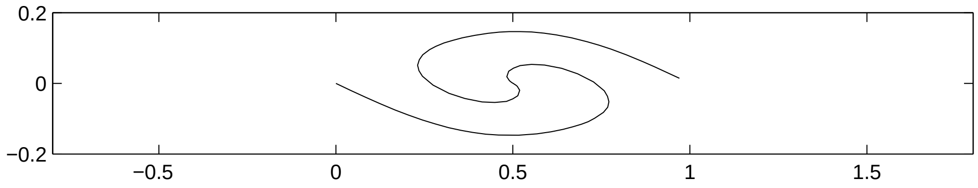
Linear Stability of flat sheet ($y \equiv 0$) solution



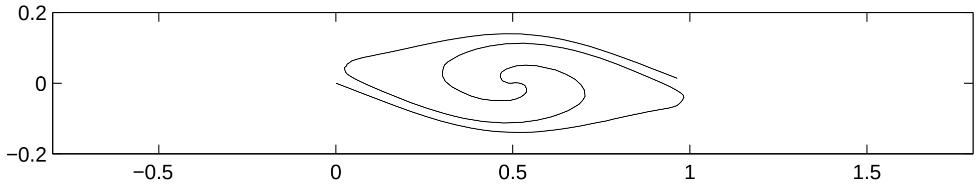
(a) With the discretized LAE (b) With Krasny's δ

- $\sigma_{max} = 0.15/\alpha, \quad k_{max} \approx 0.11/\alpha$ — length scale
- Finite thickness vorticity layer — $\sigma(k) = 0$ for $k > k_{cutoff}$
- LAE: $\sigma(k) \to 0$ asymptotically as $k \to \infty$. Similar to viscosity in Rayleigh-Taylor (Bellman and Pennington, 1954), Krasny's δ.

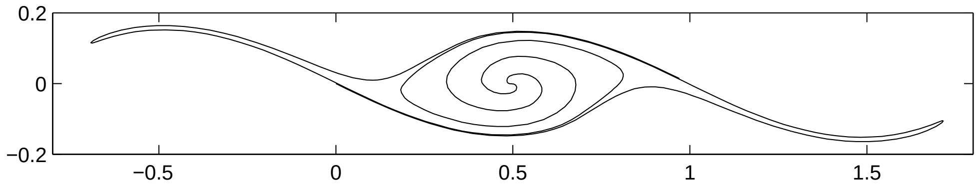
Nonlinear growth, $\alpha = 0.4$



(c) t=10



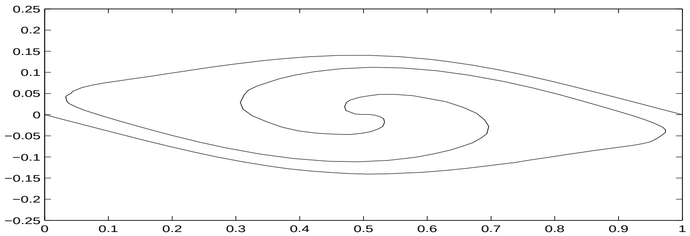
(d) t=15



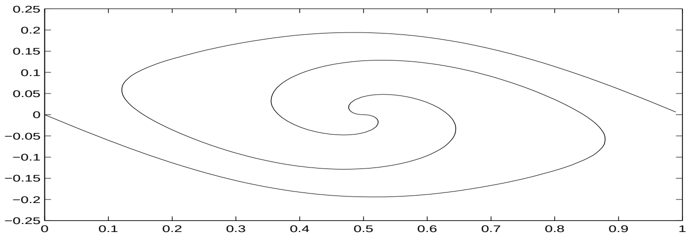
(e) t=22

KH roll up, non-uniform strain, regions of high curvature. Core stabilizes at a certain size (depending of α)

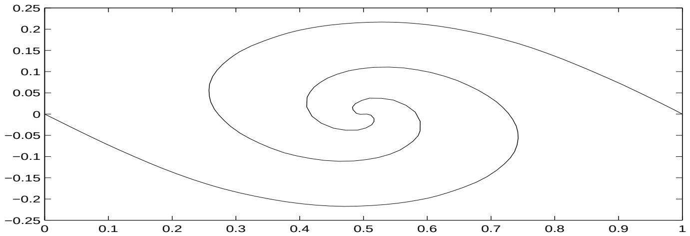
Size of structures grows as $\log \alpha$



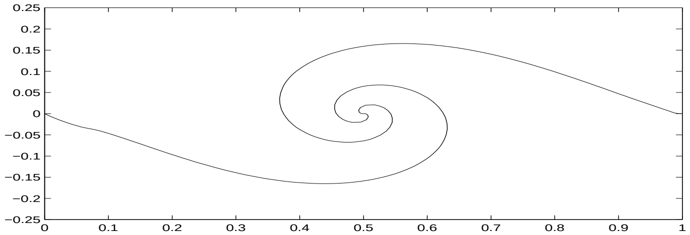
(f) $\alpha = 0.4, \quad t/\alpha = 36.95$



(g) $\alpha = 0.2, \quad t/\alpha = 29.85$



(h) $\alpha = 0.1, \quad t/\alpha = 28.4$



(i) $\alpha = 0.05, \quad t/\alpha = 30.4$

Plans

- Investigate convergence to Euler as $\alpha \to 0$ for vortex sheet
  - For $t < t_c$
  - continuous vorticity distributions in $\mathbb{R}^2$, Oliver and Shkoller, 2001
- Numerical study of this limit
  - Similar to Krasny's study for δ
  - Krasny, convergence for *all* times $(N \to \infty,\ \delta \to 0)$
- Investigate the dependence of structure size on α
- Integrability of N-vortex problems
- Consider other manifolds
- Feasibility as a control model
- Exact expressions