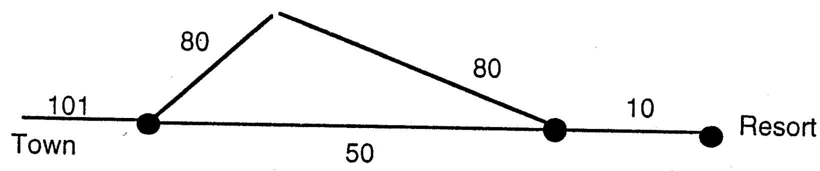
Group A: select at least two questions.

Probability Question

In the map shown below, there are several ways to get to a ski resort from the nearby town.

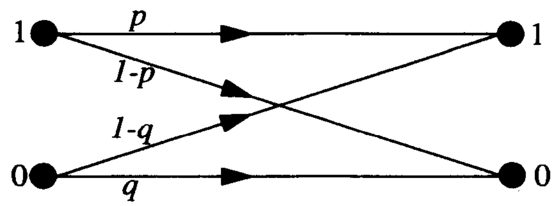


The possible routes are (1) Highway 101 to highway 50 to highway 10, which terminates at the resort. (2) Highway 101 to highway 80 to highway 10. The probability that 101 is closed due to snow is .1. The probability that 80 is closed due to snow is .25. The probability that 50 is closed due to snow is .2, and the probability that 10 is closed is .4.

Find the probability that one or more routes from the town to the ski resort is open. Assume that all roads are open or closed *independently*.

Probability

This problem has two independent parts. Show your reasoning clearly: correct answers without supporting arguments will receive no credit.



1. The asymmetric binary channel represented graphically above has the following meaning: if the source sends a "1", it has probability p of being received as a "1" and probability 1-p of being received as a "0"; If the source sends a "0" it has probability q of being received as a "0" and probability 1-q of being received as a "1". A "1" and "0" are equally probable of being sent.

   a) Find the probability that a "0" has been received

   b) Find the probability that the received bit is not the transmitted bit.

2. The jointly-distributed random variables X and Y have constant probability density function (pdf) $f_{X,Y}(x, y) = c$ over the triangle $x \geq 0$, $y \geq 0$ and $x + y \leq 1$.

   a) Find the marginal density $f_X(x)$.

   b) Find $E[X]$.

   c) Find the conditional pdf $f_{X|Y}(x|y)$.

   d) Find the conditional expectation $E[X|Y]$.

   e) Determine if X and Y are independent.

PROBABILITY.

QUESTION 1.

(a) $P(0 \text{ REC}) = P(0 \text{ REC} \mid 1 \text{ SENT})\, P(1 \text{ SENT}) + P(0 \text{ REC} \mid 0 \text{ SENT})\, P(0 \text{ S}$

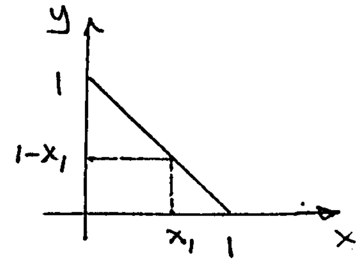
$$= (1-p)\tfrac{1}{2} + q\tfrac{1}{2} = \tfrac{1}{2}(1-p+q)$$

(b) $P(\text{ERROR}) = P(1 \text{ SENT AND } 0 \text{ REC}) + P(0 \text{ SENT AND } 1 \text{ REC})$

$$= P(0 \text{ REC} \mid 1 \text{ SENT}).P(1 \text{ SENT}) + P(1 \text{ REC} \mid 0 \text{ SENT}).P(0 \text{ SENT})$$

$$= (1-p)\tfrac{1}{2} + (1-q)\tfrac{1}{2} = \tfrac{1}{2}(2-p-q)$$

QUESTION 2



Area of triangle $= \frac{1}{2}$

so $f_{X,Y}(x,y) = 2 = c$

on this triangle.

(a) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^{1-x} 2\,dy = 2(1-x)\ , \quad x \in [0,1$

(b) $E[X] = \int_0^1 x\, 2(1-x)\,dx = 2\int_0^1 (x-x^2)\,dx = 2\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{1}{3}$

(c) $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)} = \dfrac{2}{2(1-y)} = \dfrac{1}{1-y}\ , \quad x \in [0,1$

since, by symmetry and (a), $f_Y(y) = 2(1-y)$, $y \in [0,$

(d) $E(X|Y) = \int_0^{1-y} x \cdot \frac{1}{1-y} \cdot dx = \dfrac{(1-y)^2}{2(1-y)} = \dfrac{1-y}{2}$

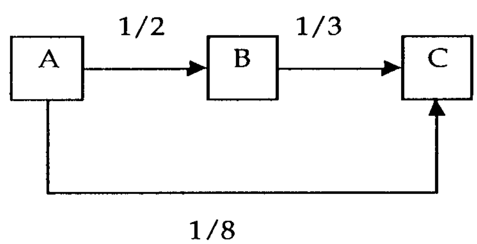
(e) $f_{X,Y}(x,y) \neq f_X(x).f_Y(y)$ so NOT independent.

Probability Question:

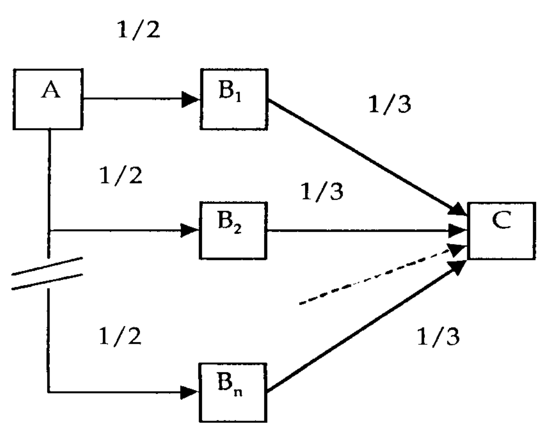
(a) A computer network consists of a source node A, destination node C, and an intermediate node B as shown in the picture below. Packets are sent from source A via B to C, and simultaneously by a direct link to C. The probabilities of successful packet transmissions are as follows

Pr(Successful Transmission A to B) = 1/2
Pr(Successful Transmission B to C) = 1/3
Pr(Successful Transmission A to C) = 1/8

Find the overall probability of the event that a packet is successfully transmitted from A to C by either path. All transmission errors are assumed independent.



(b) Consider a second network, where a packet is simultaneously transmitted to n parallel nodes B_1 through B_n, which then simultaneously transmit the packets to node C. Find the overall probability of successful transmission from A to C by any path. Note that Pr(Successful transmission A to B_i) = 1/2, for i = 1,2,...,n, and Pr(Successful transmission from B_i to C) = 1/3 for i = 1,2,...,n. Again, all transmission errors are assumed independent.



Probability

Answer:

(a) Let S be the event of overall successful transmission. Let AB be the event that a packet is successfully transmitted from A to B, with probability 1/2. Similarly P(BC) = 1/3 and P(AC) = 1/8. Let ABC be the event that a packet is successfully transmitted first from A to B, then from B to C. Then

$$P(S) = P(AC \cup ABC) = P(AC) + P(ABC) - P(AC \cap ABC)$$
$$= 1/8 + 1/6 - 1/48 = 13/48$$

(b) Again, let AB_I be the event of successful transmission from A to B_I, and B_IC be the event of successful transmission from B_I to C. It is easier to consider the probability that the packet is not successfully transmitted.

$$P(S) = 1 - \prod_{i=1}^{n}(1 - P(AB_iC))$$
$$= 1 - \prod_{i=1}^{n}(1 - 1/6)$$
$$= 1 - (1 - 1/6)^n$$

Probability Question

For full credit, you must show the reasoning behind all your answers. Parts (a) and (b) of the problem are independent.

(a) From a city's accident statistics, we know that the number of traffic accidents on a rainy day can be modeled as a Poisson random variable with mean $\log 100 \approx 4.6$, and the number of traffic accidents on a dry day can be modeled as a Poisson random variable with mean $\log 10 \approx 2.3$. (Here log denotes natural log.)

(i) On December 15, 2002, an 80 % chance of rain was predicted. Using this information and the accident statistics, find the probability that there were no accidents on that day.

(ii) You are now informed that there were indeed no accidents on December 15, 2002. Find the probability that it rained on December 15, 2002.

(b) Two bored graduate students are rolling a pair of fair dice in the lab. Whoever is the first to roll a 7 (i.e., the sum of the numbers on the two dice is 7) wins. If the game continues until someone wins, what is the probability that the first person to roll wins?

Solns to Probability Question

(a) (i) $P(\text{no accidents}) = P(\text{no accidents} \mid \text{rain})\, P(\text{rain})$

$+ P(\text{no accidents} \mid \text{dry})\, P(\text{dry})$

$= e^{-\log 100} \times 0.8 + e^{-\log 10} \times 0.2$

$= .008 + .02 = \boxed{0.028}$

(ii) $P(\text{rain} \mid \text{no accidents}) = \dfrac{P(\text{no accidents} \mid \text{rain})\, P(\text{rain})}{P(\text{no accidents})}$

$= \dfrac{e^{-\log 100} \times 0.8}{0.028} = \dfrac{.008}{.028} = 8/28$

$= \boxed{2/7}$

(b) Roll a 7 in 6 ways [(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)]

$\Rightarrow P(\text{roll } 7) = 6/36 = 1/6 = p$

A rolls first, B second $\Rightarrow$ A wins if success on odd roll

$$P(A \text{ wins}) = \underset{\text{first roll}}{\underset{\uparrow}{p}} + (1-p)^2 \underset{\text{3rd roll}}{\underset{\uparrow}{p}} + (1-p)^4 \underset{\text{5th roll}}{\underset{\uparrow}{p}} + \cdots$$

$$= p \sum_{i=0}^{\infty} \{(1-p)^2\}^i = \frac{p}{1-(1-p)^2} = \frac{1/6}{1-(5/6)^2} = \boxed{\frac{6}{11}}$$

D

Probability Question SO3

1. Consider two *independent*, discrete-valued random variables x and y. The probability mass function of each variable is

$$\Pr(\tilde{x} = k) = c_1\alpha^k, \quad k = 0,1,\ldots,\infty$$
$$\Pr(\tilde{y} = k) = c_2\beta^k, \quad k = 0,1,\ldots,\infty$$

where c_1 and c_2 are constants. It is assumed that $0 < \alpha < 1,\ 0 < \beta < 1$.

a.) Find the values of c_1 and c_2 such that Pr(x=k), Pr(y=k) are valid probability mass functions.

b.) Define a new random variable $\tilde{z} = \tilde{x} + \tilde{y}$. Find the probability mass function $\Pr(\tilde{z} = k), k = 0,1,\ldots,\infty$ in terms of α and β. Express in as compact a form as possible.

c.) Redefine $\tilde{w} = \tilde{x} / \tilde{y}$. Find $\Pr(\tilde{w} = k)$ for finite integer values of k = 0,1,... only, and again express in a compact a form as possible. Note that the event y = 0 does not contribute to $\Pr(\tilde{w} = k)$ for finite k.

Probability

Answers:

1. a.) $\sum_{0}^{\infty} \Pr(\tilde{x}=k)=1 \Rightarrow c_1 \sum_{0}^{\infty} \alpha^k = 1 \Rightarrow c_1 \frac{1}{1-\alpha}=1$

Hence $c_1 = 1-\alpha$, $c_2 = 1-\beta$.

b.)

$$\Pr(\tilde{z}=k)=\sum_{l=0}^{\infty}\Pr(\tilde{x}=l)\Pr(\tilde{y}=k-l)=c_1c_2\sum_{l=0}^{k}\alpha^l\beta^{k-l}$$

$$=c_1c_2\beta^k\sum_{l=0}^{k}(\alpha/\beta)^l=(1-\alpha)(1-\beta)\beta^k\frac{1-(\alpha/\beta)^{k+1}}{1-(\alpha/\beta)}$$

$$=\frac{(1-\alpha)(1-\beta)}{(\beta-\alpha)}\beta^{k+1}+\frac{(1-\alpha)(1-\beta)}{(\alpha-\beta)}\alpha^{k+1}, k=0,1,\ldots$$

c.)

$$\Pr(\tilde{w}=k)=\sum_{l=1}^{\infty}\Pr(\tilde{x}=kl)\Pr(\tilde{y}=l)$$

$$=(1-\alpha)(1-\beta)\left(\sum_{l=0}^{\infty}\alpha^{kl}\beta^l-1\right)$$

$$=(1-\alpha)(1-\beta)\left(\frac{1}{1-\alpha^k\beta}-1\right)$$

$$=(1-\alpha)(1-\beta)\frac{\alpha^k\beta}{1-\alpha^k\beta}$$

10|16

Fall 1996 PhD Screening Exam
Solution of Probability Problem

A roll of n coins contains one coin that has two heads. The other $n-1$ coins are fair coins. A coin is selected from the roll at random and tossed repeatedly. The first m throws are all heads.

a) What is the probability that the next toss will be tails?

b) Given fixed n, how large should m be for this observation (m heads) to imply that, with probability greater than 1/2, the tossed coin is two-headed?

Solution:

Denote events: H_m - all m tosses are heads, U - unfair (two-headed) coin is selected, F - fair coin is selected, T - next toss is tails. Let us first derive the a posteriori probability that the two headed coin was selected:

$$P(U|H_m) = \frac{P(U,H_m)}{P(H_m)}.$$

Clearly,

$$P(U,H_m) = \frac{1}{n},$$

$$P(H_m) = P(U,H_m) + P(F,H_m) = \frac{1}{n} + \frac{n-1}{n}2^{-m}$$

and hence,

$$P(U|H_m) = \frac{1}{1+(n-1)2^{-m}}.$$

a) Tails is possible only if a fair coin was selected, and

$$P(T|H_m) = \frac{P(F|H_m)}{2} = \frac{1-P(U|H_m)}{2} = \frac{(n-1)2^{-m-1}}{1+(n-1)2^{-m}}$$

b) We require

$$P(U|H_m) = \frac{1}{1+(n-1)2^{-m}} > \frac{1}{2}$$

and obtain the condition

$$m > \log_2(n-1).$$

Fall 1999 PhD Screening Exam
Solution of Probability Problem

A computer file is divided into n packets for transmission over a network. Each packet transmission may fail (independently of other transmissions) with probability p. In case of failure the transmitter is immediately notified and it re-transmits the packet repeatedly until it is successfully received.

a) With what probability will the n packets of the file be received in n consecutive packet transmissions?

b) Find the probability that it takes *exactly* K packet transmissions to receive the file.

c) What is the most probable number of packet transmissions needed until the file is received?

Solution:

a) Once the first packet is received we must have a sequence of $n-1$ successful transmissions. So the probability is $(1-p)^{n-1}$.

b) The event that the file is received in exactly K transmissions is equivalent to the intersection of two events: $n-1$ successful transmissions in $K-1$ trials, and successful transmission in the Kth trial:

$$P[K] = \binom{K-1}{n-1}(1-p)^{n-1}p^{K-n} \cdot (1-p) = \binom{K-1}{n-1}(1-p)^{n}p^{K-n}.$$

c) From (b),

$$P[K] = P[K-1]\frac{(K-1)p}{K-n}$$

So $P[K]$ is increasing as long as

$$\frac{(K-1)p}{K-n} > 1,$$

or

$$K < \frac{n-p}{1-p}$$

Hence, the most probable K is

$$\hat{K} = \lfloor \frac{n-p}{1-p} \rfloor.$$

Fall 2002 PhD Screening Exam
Probability

Let $X_1, X_2, \ldots, X_n$ be independent identically distributed continuous random variables with probability density function $f_X(x)$, cumulative distribution function $F_X(x)$, mean μ and variance σ^2. We define random variables $Y = \max[X_1, X_2, \ldots, X_n]$, $Z = \min[X_1, X_2 \ldots, X_n]$ and $W = Y - Z$. *Your final answers to the following may only be expressed in terms of f_X, F_X, μ and σ.*

a) Find the density $f_Y(y)$ and distribution $F_Y(y)$. (40%)

c) Find the density $f_Z(z)$ and distribution $F_Z(z)$. (40%)

a) For the case $n = 2$, find the second moment $E\{W^2\}$. (20%)

Solution:

a) We first consider the cdf:

$F_Y(y) = \Pr\{Y \leq y\} = \Pr\{\max[X_1, \ldots, X_n] \leq y\} = \Pr\{X_1 \leq y, \ldots, X_n \leq y\} = F_X^n(y),$

where last equality is by independence. Now,

$$f_Y(y) = \frac{d}{dy} F_Y(y) = n f_X(y) F_X^{n-1}(y).$$

b) Consider the cdf:

$1 - F_Z(z) = \Pr\{Z > z\} = \Pr\{\min[X_1, \ldots, X_n] > z\} = \Pr\{X_1 > z, \ldots, X_n > z\},$

and hence

$$1 - F_Z(z) = [1 - F_X(z)]^n,$$

and

$$F_Z(z) = 1 - [1 - F_X(z)]^n.$$

Next,

$$f_Z(z) = \frac{d}{dz} F_Z(z) = n f_X(z)[1 - F_X(z)]^{n-1}.$$

c) We observe that $W^2 = (\max[X_1, X_2] - \min[X_1, X_2])^2 = (X_1 - X_2)^2$. Hence,

$$E\{W^2\} = E\{X_1^2\} + E\{X_2^2\} - 2E\{X_1 X_2\} = 2(\mu^2 + \sigma^2) - 2\mu^2 = 2\sigma^2,$$

where we made use of the independence of X_1 and X_2.