Problem 4.7.8 Solution

The joint PDF of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} (x+y)/3 & 0 \le x \le 1, 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Before calculating moments, we first find the marginal PDFs of X and Y. For $0 \le x \le 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^2 \frac{x+y}{3}\,dy = \left.\frac{xy}{3} + \frac{y^2}{6}\right|_{y=0}^{y=2} = \frac{2x+2}{3} \tag{2}$$

For $0 \le y \le 2$,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_0^1 (\frac{x}{3} + \frac{y}{3})\,dx = \left.\frac{x^2}{6} + \frac{xy}{3}\right|_{x=0}^{x=1} = \frac{2y+1}{6} \tag{3}$$

Complete expressions for the marginal PDFs are

$$f_X(x) = \begin{cases} \frac{2x+2}{3} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} \frac{2y+1}{6} & 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(a) The expected value of X is

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 x\frac{2x+2}{3}\,dx = \left.\frac{2x^3}{9} + \frac{x^2}{3}\right|_0^1 = \frac{5}{9} \tag{5}$$

The second moment of X is

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_0^1 x^2\frac{2x+2}{3}\,dx = \left.\frac{x^4}{6} + \frac{2x^3}{9}\right|_0^1 = \frac{7}{18} \tag{6}$$

The variance of X is $\text{Var}[X] = E[X^2] - (E[X])^2 = 7/18 - (5/9)^2 = 13/162$.

(b) The expected value of Y is

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_0^2 y\frac{2y+1}{6}\,dy = \left.\frac{y^2}{12} + \frac{y^3}{9}\right|_0^2 = \frac{11}{9} \tag{7}$$

The second moment of Y is

$$E\left[Y^2\right] = \int_{-\infty}^{\infty} y^2 f_Y(y)\,dy = \int_0^2 y^2\frac{2y+1}{6}\,dy = \left.\frac{y^3}{18} + \frac{y^4}{12}\right|_0^2 = \frac{16}{9} \tag{8}$$

The variance of Y is $\text{Var}[Y] = E[Y^2] - (E[Y])^2 = 23/81$.

(c) The correlation of X and Y is

$$E[XY] = \iint xy f_{X,Y}(x,y)\,dx\,dy \tag{9}$$

$$= \int_0^1 \int_0^2 xy(\frac{x+y}{3})\,dy\,dx \tag{10}$$

$$= \int_0^1 (\left.\frac{x^2y^2}{6} + \frac{xy^3}{9}\right|_{y=0}^{y=2})\,dx \tag{11}$$

$$= \int_0^1 (\frac{2x^2}{3} + \frac{8x}{9})\,dx = \left.\frac{2x^3}{9} + \frac{4x^2}{9}\right|_0^1 = \frac{2}{3} \tag{12}$$

The covariance is $\text{Cov}[X,Y] = E[XY] - E[X]E[Y] = -1/81$.

(d) The expected value of X and Y is

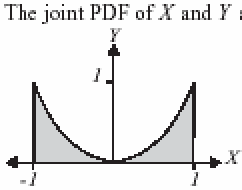
$$E[X+Y] = E[X] + E[Y] = 5/9 + 11/9 = 16/9 \tag{13}$$

(e) By Theorem 4.15,

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] + 2\,\text{Cov}[X,Y] = \frac{13}{162} + \frac{23}{81} - \frac{2}{81} = \frac{55}{162} \tag{14}$$

Problem 4.7.10 Solution

The joint PDF of X and Y and the region of nonzero probability are



$$f_{X,Y}(x,y) = \begin{cases} 5x^2/2 & -1 \le x \le 1, 0 \le y \le x^2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) The first moment of X is

$$E[X] = \int_{-1}^{1}\int_{0}^{x^2} x\frac{5x^2}{2}\,dy\,dx = \int_{-1}^{1}\frac{5x^5}{2}\,dx = \left.\frac{5x^6}{12}\right|_{-1}^{1} = 0 \tag{2}$$

Since $E[X] = 0$, the variance of X and the second moment are both

$$\text{Var}[X] = E[X^2] = \int_{-1}^{1}\int_{0}^{x^2} x^2\frac{5x^2}{2}\,dy\,dx = \left.\frac{5x^7}{14}\right|_{-1}^{1} = \frac{10}{14} \tag{3}$$

(b) The first and second moments of Y are

$$E[Y] = \int_{-1}^{1}\int_{0}^{x^2} y\frac{5x^2}{2}\,dy\,dx = \frac{5}{14} \tag{4}$$

$$E[Y^2] = \int_{-1}^{1}\int_{0} x^2y^2\frac{5x^2}{2}\,dy\,dx = \frac{5}{26} \tag{5}$$

Therefore, Y has variance

$$\text{Var}[Y] = \frac{5}{26} - \left(\frac{5}{14}\right)^2 = .0576 \tag{6}$$

(c) Since $E[X] = 0$, $\text{Cov}[X,Y] = E[XY] - E[X]E[Y] = E[XY]$. Thus,

$$\text{Cov}[X,Y] = E[XY] = \int_{1}^{1}\int_{0}^{x^2} xy\frac{5x^2}{2}\,dy\,dx = \int_{-1}^{1}\frac{5x^7}{4}\,dx = 0 \tag{7}$$

(d) The expected value of the sum $X + Y$ is

$$E[X+Y] = E[X] + E[Y] = \frac{5}{14} \tag{8}$$

(e) By Theorem 4.15, the variance of $X + Y$ is

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] + 2\,\text{Cov}[X,Y] = 5/7 + 0.0576 = 0.7719 \tag{9}$$

Problem 6.2.2 Solution

The joint PDF of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} 1 & 0 \le x, y \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Proceeding as in Problem 6.2.1, we must first find $F_W(w)$ by integrating over the square defined by $0 \le x, y \le 1$. Again we are forced to find $F_W(w)$ in parts as we did in Problem 6.2.1 resulting in the following integrals for their appropriate regions. For $0 \le w \le 1$,

$$F_W(w) = \int_0^w \int_0^{w-x} dx\, dy = w^2/2 \tag{2}$$

For $1 \le w \le 2$,

$$F_W(w) = \int_0^{w-1} \int_0^1 dx\, dy + \int_{w-1}^1 \int_0^{w-y} dx\, dy = 2w - 1 - w^2/2 \tag{3}$$

The complete expression for the CDF of W is

$$F_W(w) = \begin{cases} 0 & w < 0 \\ w^2/2 & 0 \le w \le 1 \\ 2w - 1 - w^2/2 & 1 \le w \le 2 \\ 1 & \text{otherwise} \end{cases} \tag{4}$$

With the CDF, we can find $f_W(w)$ by differentiating with respect to w.

$$f_W(w) = \begin{cases} w & 0 \le w \le 1 \\ 2 - w & 1 \le w \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Problem 6.2.3 Solution

By using Theorem 6.5, we can find the PDF of $W = X + Y$ by convolving the two exponential distributions. For $\mu \ne \lambda$,

$$\begin{aligned} f_W(w) &= \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)\, dx && (1) \\ &= \int_0^w \lambda e^{-\lambda x} \mu e^{-\mu(w-x)}\, dx && (2) \\ &= \lambda \mu e^{-\mu w} \int_0^w e^{-(\lambda-\mu)x}\, dx && (3) \\ &= \begin{cases} \frac{\lambda\mu}{\lambda-\mu}\left(e^{-\mu w} - e^{-\lambda w}\right) & w \ge 0 \\ 0 & \text{otherwise} \end{cases} && (4) \end{aligned}$$

When $\mu = \lambda$, the previous derivation is invalid because of the denominator term $\lambda - \mu$. For $\mu = \lambda$, we have

$$\begin{aligned} f_W(w) &= \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)\, dx && (5) \\ &= \int_0^w \lambda e^{-\lambda x} \lambda e^{-\lambda(w-x)}\, dx && (6) \\ &= \lambda^2 e^{-\lambda w} \int_0^w dx && (7) \\ &= \begin{cases} \lambda^2 w e^{-\lambda w} & w \ge 0 \\ 0 & \text{otherwise} \end{cases} && (8) \end{aligned}$$

Note that when $\mu = \lambda$, W is the sum of two iid exponential random variables and has a second order Erlang PDF.

Problem 10.2.1 Solution

- In Example 10.3, the daily noontime temperature at Newark Airport is a discrete time, continuous value random process. However, if the temperature is recorded only in units of one degree, then the process was would be discrete value.
- In Example 10.4, the the number of active telephone calls is discrete time and discrete value.
- The dice rolling experiment of Example 10.5 yields a discrete time, discrete value random process.
- The QPSK system of Example 10.6 is a continuous time and continuous value random process.

Problem 10.2.4 Solution

The statement is *false*. As a counterexample, consider the rectified cosine waveform $X(t) = R|\cos 2\pi ft|$ of Example 10.9. When $t = \pi/2$, then $\cos 2\pi ft = 0$ so that $X(\pi/2) = 0$. Hence $X(\pi/2)$ has PDF

$$f_{X(\pi/2)}(x) = \delta(x) \quad (1)$$

That is, $X(\pi/2)$ is a discrete random variable.

B. From the textbook, two random variables X and Y are orthogonal if their correlation, $r_{X,Y}=0$.

Thus, we want $r_{V,W} = 0$;

$$r_{V,W} = E[VW]$$
$$r_{V,W} = E[(X+aY)(X-aY)]$$
$$r_{V,W} = E[X^2 + aXY - aXY - a^2Y^2]$$
$$r_{V,W} = E[X^2] - a^2E[Y^2]$$

Now we set $r_{V,W} = 0$

$$E[X^2] - a^2E[Y^2] = 0$$

And solve for a:

$$a = \pm\sqrt{\left(\frac{E[X^2]}{E[Y^2]}\right)}$$