MAT 202.

INTRODUCTION TO MATHEMATICS FOR DSP.

SET 1: SIGNALS & SYSTEMS BASICS.

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Signals describe a variety of physical phenomenon e.g. speech & music, images, video, seismic signals, ECG and so on.

Signals are represented mathematically as functions of one or more independent variables.

e.g. 1) speech signal is represented mathematically by acoustic pressure as a function of time.

2) a picture is represented by brightness as a function of two spatial variables.

I] CLASSIFICATION OF SIGNALS

A. CONTINUOUS TIME & DISCRETE TIME

Continuous time signals are those in which the independent variable is continuous, and the signals are defined for a continuum of values of the independent variable.

We use the symbol 't' to denote the independent variable for continuous time signals and we enclose it in paranthesis (·)

e.g. $x(t)$

Discrete time signals are those in which the independent variable takes a discrete set of values, and the signals are defined for a discrete set of values of the independent variable.

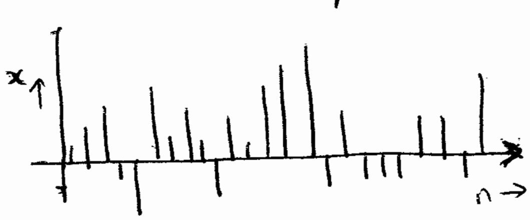
We use the symbol 'n' to denote the independent variable for discrete time signals and we enclose it in paranthesis [ ]

e.g. $x[n]$

B. STOCHASTIC & DETERMINISTIC

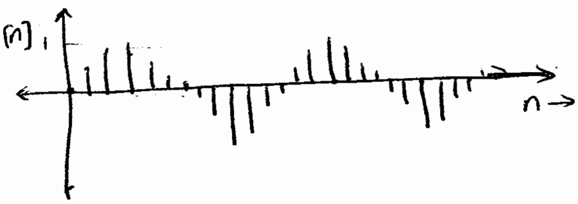
Stochastic signals cannot be characterized by a simple, well-defined mathematical equation and their future values cannot be predicted

e.g.



Deterministic signals have fixed values and can be represented using a mathematical expression, rule or a table. These signals are relatively easy to analyze, and we can make accurate assumptions about their past and future behaviour.

e.g: $x[n] = \sin(0.5n)$



C. PERIODIC & APERIODIC SIGNALS:

Periodic signals repeat with some period N. They satisfy the following mathematical equation

$x[n] = x[n+kN]$ where $k = 1, 2, \ldots$

$N \rightarrow$ fundamental period.

The fundamental period is the smallest value of N, that allows the equation

$x[n] = x[n+N]$

to be true.

e.g:

$N = 5$

$x[5] = x[0+5] = x[0]$

$x[6] = x[1+5] = x[1]$

$\vdots$

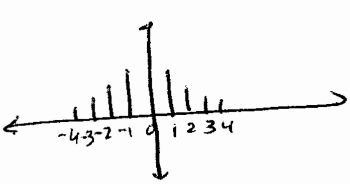
Signals which are not periodic i.e which do not satisfy the condition for periodicity, are said to be aperiodic or non-periodic.

D. EVEN & ODD SIGNALS:

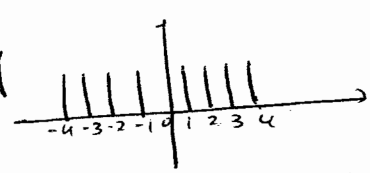
An even signal is symmetric about the vertical axis. It satisfies the equation

$x[n] = x[-n]$

e.g



e.g



$x[4] = x[-4]$

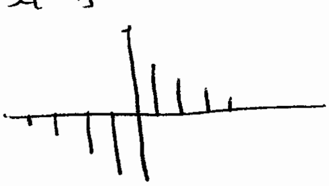
$x[3] = x[-3]$

$\vdots$

An odd signal is 'anti-symmetric' about the vertical axis. It satisfies the equation

$x[n] = -x[-n]$

e.g:



e.g

$x[4] = -x[-4]$

$x[3] = -x[-3]$

$\vdots$

$x[0] = 0$ {always} for odd signals.

E] ANALOG & DIGITAL SIGNALS

Analog corresponds to continuous y-axis. Thus, the signal can take on any values within the range provided.

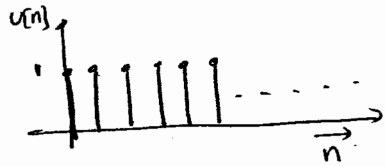
Digital corresponds to discrete y-axis. Thus, the signal can take on only a discrete set of values in the range provided.

Continuous time & discrete-time signals can be both analog or digital. Generally, digital usually implies discrete time as well.

II] COMMON SIGNAL FUNCTIONS:

A] UNIT STEP FUNCTION

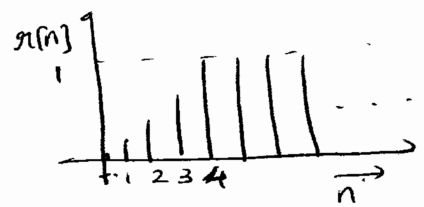
$$u[n] = 0 \quad n < 0$$
$$= 1 \quad n \geq 0$$



B] RAMP FUNCTION

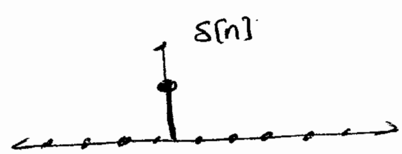
$$r[n] = 0 \quad n < 0$$
$$= \frac{n}{N_0} \quad 0 \leq n \leq N_0$$
$$= 1 \quad n > N_0$$

$N_0 = 4$



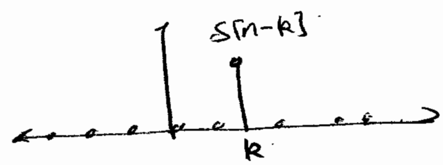
C] UNIT IMPULSE FUNCTION

$$\delta[n] = 1 \quad n \neq 0$$
$$0 \quad n \neq 0$$



The unit impulse function delayed by k-samples is given by

$$\delta[n-k] = 1 \quad n = k$$
$$0 \quad n \neq k$$



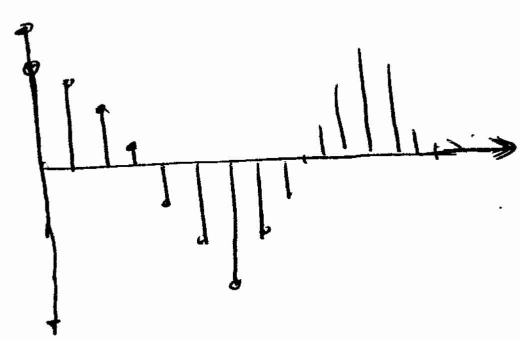
D] SINUSOIDAL FUNCTION

$$x[n] = A\cos(\omega_0 n + \phi)$$

$A \to$ amplitude

$\omega_0 \to$ angular freq $= \frac{2\pi}{N}$

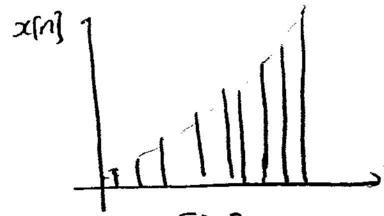
$\phi \to$ phase



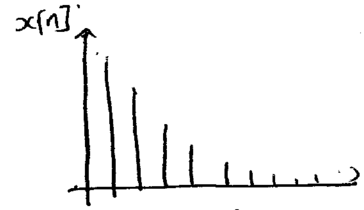
E] EXPONENTIAL FUNCTION

$x[n] = A\alpha^{\sigma n}$

$\sigma > 0 \rightarrow$ rising exponential
$\sigma < 0 \rightarrow$ decaying exponential



$\sigma > 0$
rising exponential



$\sigma < 0$
decaying exponential

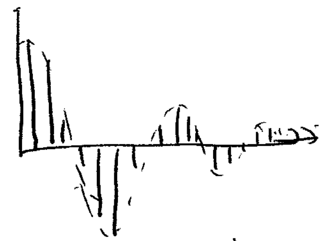
F] COMPLEX EXPONENTIAL FUNCTION

$x[n] = A\alpha^{n}$

where $\alpha = e^{(\sigma + j\omega)}$

$\sigma > 0 \rightarrow$ rising exponential
$\sigma < 0 \rightarrow$ decaying exponential

$\sigma > 0$



$\sigma < 0$

* We will come back to this, after covering complex numbers.

III] PROPERTIES OF A SIGNAL

A] SIGNAL ENERGY

$$E = \sum_{n=0}^{\infty} (x[n])^2$$

B] SIGNAL POWER

a) For periodic signals

$$P = \frac{1}{2N+1} \sum_{n=-N}^{N} (x[n])^2$$

b) For non-periodic signals

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} (x[n])^2$$

An infinite energy signal with finite average power is called a POWER SIGNAL.
e.g. periodic signals

A finite energy signal with zero average power is called an ENERGY SIGNAL.
e.g. finite-length signals.

IV Basic Operations On Signals

A] Modulation:

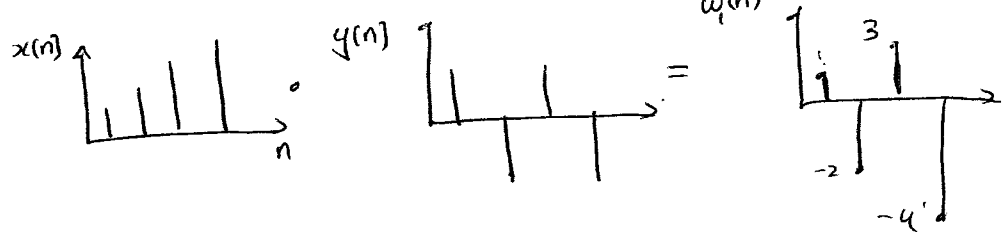
If $x[n]$ and $y[n]$ are two known sequences, then the sample by sample product of the two sequences is referred to as modulation

$w_1[n] = x[n] \cdot y[n]$

e.g $x[n] = \{1, 2, 3, 4\}$

$y[n] = \{1, -1, 1, -1\}$

$\Rightarrow w_1[n] = \{1, -2, 3, -4\}$



B] Addition

If $x[n]$ and $y[n]$ are 2 known sequences, then the sample by sample addition of the two sequences is given by

$w_2[n] = x[n] + y[n]$

e.g: $x[n] = \{1, 2, 3, 4\}$

$y[n] = \{1, -1, 1, -1\}$

$w_2[n] = \{2, 1, 4, 3\}$

C] Scalar Multiplication

A new sequence is generated by multiplying each sample of the original sequence by a scalar A.

$w_3[n] = A x[n]$

e.g: $x[n] = \{1, 2, 3, 4\}$

$A = 1.5$

$w_3[n] = \{1.5, 3, 4.5, 6\}$

D] Time Shifting:

$w_4[n] = x[n-N]$

if $N > 0 \rightarrow$ delaying operation

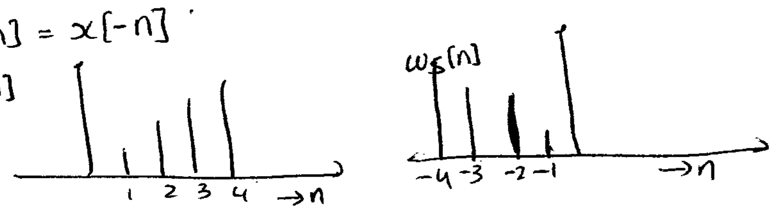
$N < 0 \rightarrow$ advancing operation

e.g: $x[n] = \{1, 2, 3, 4\}$

E] Time Reversal

$w_5[n] = x[-n]$

e.g $x[n]$



F] TIME SCALING

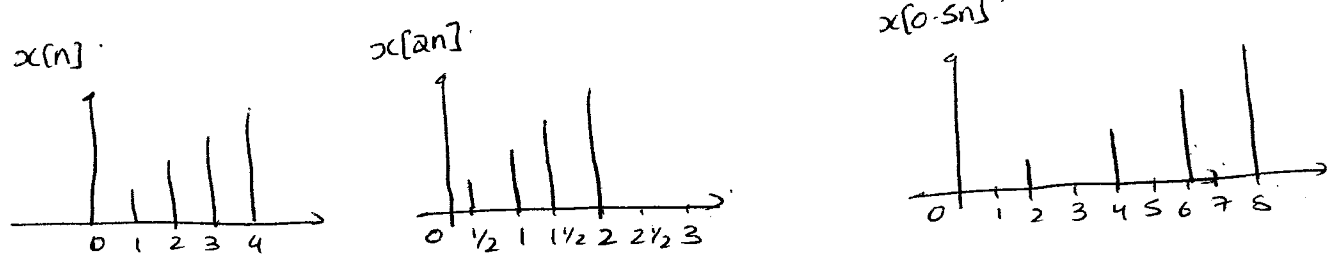
Time scaling compresses or dilates the signal by multiplying the time variable by some amount.

$w_6[n] = x[an]$

if $a > 1 \rightarrow$ compression

$a < 1 \rightarrow$ dilation

e.g



V] SYSTEMS & PROPERTIES OF SYSTEMS

A system can be viewed as a process in which the input signals are transformed by the system, or the system responds in some way to the input, resulting in other signals at the output.

$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$

e.g: Accumulator:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]$$

e.g: Difference operator

$$y[n] = x[n] - x[n-1]$$

A] CONTINUOUS & DISCRETE SYSTEM

A system where the input & output signals are continuous is called a continuous system.

A system where the input & output signals are discrete is called a discrete system.

B] SYSTEMS WITH AND WITHOUT MEMORY

A system is said to be 'memoryless' if the output for each value of the independent variable at a given time depends only on the current value of the input at that time.

e.g $y[n] = 2x[n] + (x[n])^2$

If the output of the system depends on current & previously stored values of the input signal, it is said to be a system with memory.

e.g $y[n] = \sum_{k=-\infty}^{n} x[k]$

C] INVERTIBLE & INVERSE SYSTEMS

A system is said to be invertible if distinct inputs lead to distinct outputs.
If a system is invertible, an inverse system exists, which when cascaded with the original system, gives an output equal to the input to the first system.

e.g: $y[n] = 2x[n]$ $\quad x[n] \rightarrow \boxed{2} \rightarrow y[n]$

$x[n] = \frac{1}{2}y[n]$ $\quad x[n] \rightarrow \boxed{2} \rightarrow y[n] \rightarrow \boxed{1/2} \rightarrow w[n] = x[n]$

↑ Inverse system.

D] CAUSAL AND NONCAUSAL SYSTEMS

A system is said to be causal if its output depends on present and past values of the input only, & not on future values. Else the system is said to be non-causal.

e.g: Causal system

$$y[n] = x[n] + x[n-1]$$

e.g Non-causal system

$$y[n] = x[n] + x[n+1].$$

E] LINEAR & NON-LINEAR SYSTEMS:

A linear system obeys the properties of scaling (homogeneity) and additivity (superposition).

a) HOMOGENEITY

If $x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$.

then homogeneity implies

$ax[n] \rightarrow \boxed{h[n]} \rightarrow ay[n]$.

b) ADDITIVITY

If $x_1[n] \rightarrow \boxed{h[n]} \rightarrow y_1[n]$

and

$x_2[n] \rightarrow \boxed{h[n]} \rightarrow y_2[n]$

then $x_1[n] + x_2[n] \rightarrow \boxed{h[n]} \rightarrow y_1[n] + y_2[n]$.

Thus for a LINEAR SYSTEM

$x_1[n] \rightarrow \boxed{h[n]} \rightarrow y_1[n]$

& $x_2[n] \rightarrow \boxed{h[n]} \rightarrow y_2[n]$.

then $ax_1[n] + bx_2[n] \rightarrow \boxed{h[n]} \rightarrow ay_1[n] + by_2[n]$.

A system which does not satisfy the conditions for linearity is a non-linear system.

e.g: Linear systems

$y[n] = nx[n]$

$y[n] = ax[n]$.

Non-linear systems

$y[n] = x^2[n]$.

$y[n] = ax[n] + b$.

F] TIME VARIANT & TIME INVARIANT SYSTEM:

A system is Time invariant if the behavior & characteristics of the system are fixed over time. Specifically, a system is time invariant if a time-shift in the input signal results in an identical time shift in the output signal.

i.e if $x[n] \rightarrow \boxed{h[n]} \rightarrow y[n] \Rightarrow H\{x[n]\} = y[n]$.

then $x[n-n_0] \rightarrow \boxed{h[n]} \rightarrow y[n-n_0] \Rightarrow H\{x[n-n_0]\} = y[n-n_0]$.

If the system does not satisfy the above property, it is said to be time variant.

e.g: Time invariant

$y[n] = \sin(x[n])$

$y[n] = a\,x[n]$.

Time variant

$y[n] = n\,x[n]$.

G] STABLE & UNSTABLE SYSTEMS

A stable system is one in which the output does not diverge, as long as the input does not diverge. Such a system is said to be a BOUNDED INPUT BOUNDED OUTPUT (BIBO) SYSTEM. The condition can be mathematically expressed as:

$$|y[n]| \leq M_y < \infty \quad \forall n$$

provided $|x[n]| \leq M_x < \infty \quad \forall n$.

If the above conditions are not met, the system output may diverge even if the input is bounded, & the system is said to be UNSTABLE.

e.g: Stable system:

$y[n] = \text{constant}$

$y[n] = a\,x[n]$.

UNSTABLE SYSTEM

$y[n] = n\,x[n]$.

$y[n] = \sum_{k=-\infty}^{n} |x[n]|$