MAT202.

Introduction To Mathematics For DSP.

Set 2: Complex Numbers.

- Niranjan Shetty

A complex number is an ordered pair (x, y) of real numbers x and y, written as

$$z = (x, y)$$

x is called the real part of the complex number z $\Rightarrow x = \mathrm{Re}(z)$

y is called the imaginary part of the complex number z $\Rightarrow y = \mathrm{Im}(z)$

We need complex numbers to solve equations such as $z^2 = -1$, which have no real solutions. As we will see later on, complex numbers can be used to represent sinusoidal signals. Thus, an understanding of complex numbers is crucial to signal processing.

I] NOTATION FOR COMPLEX NUMBERS:

The two basic notations for representing complex numbers are a) rectangular or Cartesian form b) Polar form.

A] RECTANGULAR FORM:

$$\begin{aligned} z &= (x, y) \\ &= x + jy \\ &= \mathrm{Re}(z) + j \cdot \mathrm{Im}(z) \qquad \text{— (1)} \end{aligned}$$

In this notation, the complex number 'z' is expressed in terms of a real part (x) and an imaginary part (y). The symbol 'j' is used to identify the imaginary part of a complex number. The value of $j = \sqrt{-1}$.

e.g: $a + jb$; $0.5 + j1.7$

$4 + j3$; $-4 - j7$

We can represent the complex number 'z', using a rectangular (cartesian) co-ordinate system comprising of a real axis (x-axis) and imaginary axis (y-axis), perpendicular to each other.

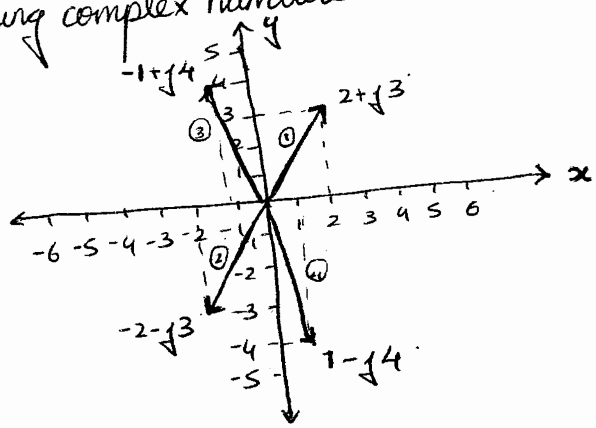
e.g: Let us plot the following complex numbers

① $2 + j3$

② $-1 + j4$

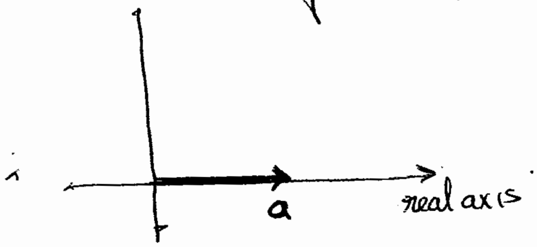
③ $-2 - j3$

④ $1 - j4$

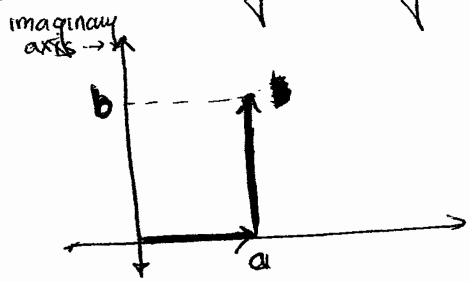


For plotting a complex number of the form $z = a + jb$. We perform the following steps.

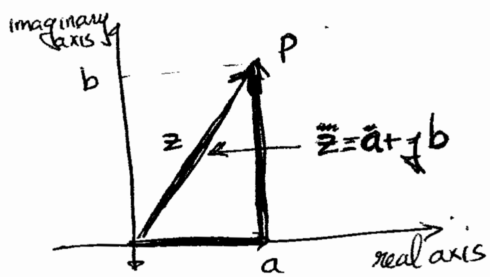
① Move a distance of a-units along the horizontal (x) axis starting from the origin.
If a is +ve, we move in the positive (right) direction horizontally & if a is -ve, we move in the negative (left) direction horizontally.



② Move a distance of b-units along the vertical (y) direction starting from point a.
If b is positive, we move along the positive (upward) direction.
If b is negative, we move along the negative (downward) direction.



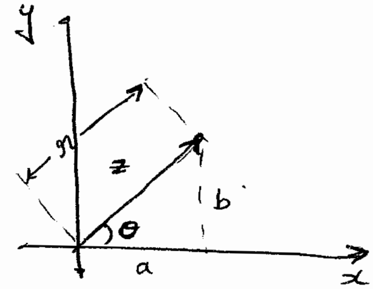
③ We now draw a vector from the origin to the final point. This represents the desired complex number $z = a + jb$.



The point (denoted by P) where the arrow ends is referred to as "the point z in the complex plane".

B] POLAR FORM.

The geometrical representation of the complex number 'z', has two important characteristics a) magnitude or length b) direction



The magnitude refers to the physical length of the vector 'z'. as shown in the figure alongside. It is denoted by 'r'. Mathematically, it can be calculated as

$$r = \sqrt{a^2 + b^2}.$$

This follows from the Pythagoras Theorem.

The direction of the vector 'z' can be quantified in terms of the angle ~~the~~ it makes with the horizontal axis (x-axis). It is denoted by θ (theta), and is mathematically calculated as

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

This follows from basic trigonometry.

In literature, ~~the~~ magnitude 'r' is often referred to as the 'ABSOLUTE VALUE' or 'MODULUS' of z ~~[illegible]~~

$$r = \sqrt{a^2+b^2} = |z| \leftarrow \text{notation for 'modulus of z'}.$$

Similarly. the angle 'θ' is referred to as the 'argument' of z.

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \arg z \leftarrow \text{notation for 'argument' of z}.$$

Thus, knowledge of the 'modulus' and 'argument' of the z is suffient to completely define the complex number z.

This gives us another way to represent the complex number z, called the polar form. It is expressed as

$$z \leftrightarrow r\angle\theta.$$

We shall see the precise expression for the polar form in the next page.

~~CONVERSIONS~~

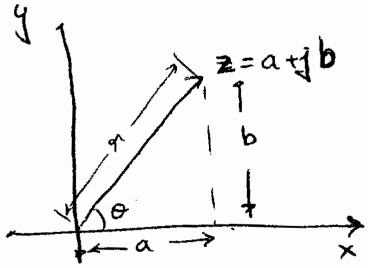
NOTE: θ can be expressed in units of radians or degrees.

The range for θ is $0 < \theta < 360°$ (in degrees)

$0 < \theta < 2\pi$ (in radians)

To convert ~~from~~ θ from degrees to radians: $\theta(\text{radians}) = \frac{\theta(\text{degrees})}{360} \times 2\pi$.

C] RELATION BETWEEN RECTANGULAR AND POLAR CO-ORDINATES.



As seen earlier, the rectangular form of the complex number 'z' is given by:

$$z = a + jb \quad -(1)$$

From the figure, we see that

$$a = r\cos\theta \quad -(2)$$

$$b = r\sin\theta \quad -(3)$$

$\therefore$ Substituting (2) & (3) in (1), we have

$$z = r\cos\theta + jr\sin\theta$$

$$= r(\cos\theta + j\sin\theta) \quad -(4)$$

We now define the complex exponential '$e^{j\theta}$' as

$$e^{j\theta} = \cos\theta + j\sin\theta.$$

This is known as EULER'S FORMULA (pronounced 'oiler')

Substituting Euler's formula in (4), we have

$$z = re^{j\theta}.$$

This is the desired expression for the polar form. The above analysis demonstrates the method for obtaining the polar form of the complex number 'z', starting with the rectangular form.

D] EULER & INVERSE EULER FORMULA.

Euler's formula for the complex exponential is given by

$$e^{j\theta} = \cos\theta + j\sin\theta \quad -(5)$$

If we replace θ with $-\theta$ in the above equation, we have

$$e^{-j\theta} = \cos\theta - j\sin\theta \quad \left\{ \because \begin{array}{l} \cos(-\theta) = \cos\theta \\ \sin(-\theta) = -\sin\theta \end{array} \right\} \quad -(6)$$

Adding (5) & (6), we get

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad -(7)$$

Substracting (6) from (5), we get

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \quad -(8)$$

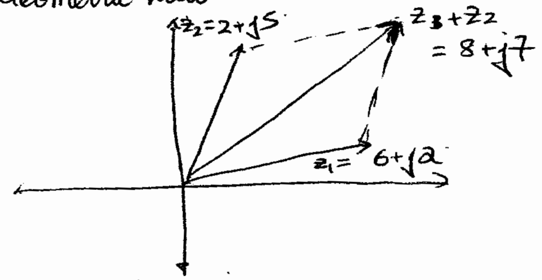
Equations (7) & (8) are called the INVERSE EULER RELATIONS.

II] OPERATIONS ON COMPLEX NUMBERS

A] ADDITION

Let $z_1 = x_1 + jy_1$ & $z_2 = x_2 + jy_2$ $\Rightarrow$ e.g. Geometric view.

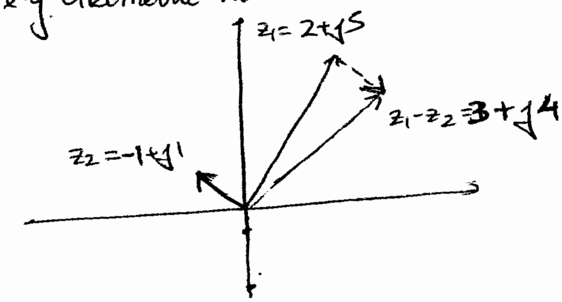
$$\therefore z_1 + z_2 = (x_1 + jy_1) + (x_2 + jy_2)$$
$$= (x_1 + x_2) + j(y_1 + y_2)$$



B] SUBTRACTION:

$$z_1 - z_2 = (x_1 + jy_1) - (x_2 + jy_2)$$
$$= (x_1 - x_2) + j(y_1 - y_2)$$

$\Rightarrow$ e.g. Geometric view



C] MULTIPLICATION

$$z_1 \cdot z_2 = (x_1 + jy_1)(x_2 + jy_2)$$
$$= x_1x_2 + j^2 y_1y_2 + j(x_1y_2 + x_2y_2)$$

Since $j = \sqrt{-1}$
$j^2 = -1$

$$\therefore z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + j(x_1y_2 + x_2y_1)$$

The above represents multiplication for the rectangular representation. Often, it is easier to perform multiplication in the polar form.

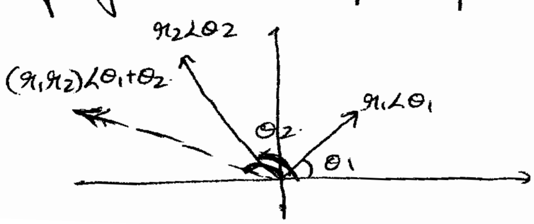
Let $z_1 = r_1 e^{j\theta_1}$
$z_2 = r_2 e^{j\theta_2}$

$$\therefore z_1 \cdot z_2 = r_1 e^{j\theta_1} \cdot r_2 e^{j\theta_2}$$
$$= (r_1 r_2) e^{j(\theta_1 + \theta_2)}$$

Thus, multiplication of two complex numbers involves multiplication of their modulus & addition of their arguments.

If we have to multiply a large number of complex numbers, it is easier to do so using the polar form, than using the rectangular form.

e.g: Geometric view of multiplication



D] DIVISION

If $z_1 = x_1 + jy_1$ & $z_2 = x_2 + jy_2$; then

$$\frac{z_1}{z_2} = \frac{x_1+jy_1}{x_2+jy_2} = \frac{(x_1+jy_1)}{(x_2+jy_2)}\,\frac{(x_2-jy_2)}{(x_2-jy_2)} = \frac{(x_1+jy_1)(x_2-jy_2)}{x_2^2+y_2^2}$$

$$= \underbrace{\frac{(x_1x_2+y_1y_2)}{x_2^2+y_2^2}}_{\text{real part}} + j\underbrace{\frac{(x_2y_1-x_1y_2)}{x_2^2+y_2^2}}_{\text{imaginary part}}$$

(x_2-jy_2) is the complex conjugate of (x_2+jy_2). In case of division, we multiply both the numerator & denominator, so as to separate the final result into real & imaginary parts as demonstrated above.

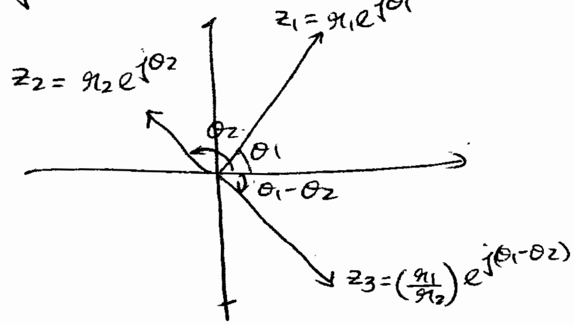
Just as in multiplication, division is easier performed using the polar form of the complex number, as shown below.

$$\frac{z_1}{z_2} = \frac{r_1e^{j\theta_1}}{r_2e^{j\theta_2}} = \underbrace{\left(\frac{r_1}{r_2}\right)}_{\text{modulus}} e^{\overbrace{j(\theta_1-\theta_2)}^{\text{argument}}} = r_3e^{j\theta_3} = z_3 \text{ (say)}$$

$\therefore r_3 = \frac{r_1}{r_2}$

$\theta_3 = \theta_1 - \theta_2$.

e.g: Geometric view



E] COMPLEX CONJUGATION

If $z = x+jy$, then the complex conjugate of z is denoted as $\bar{z}$ or z^* and is expressed as

$\bar{z} = x - jy$.

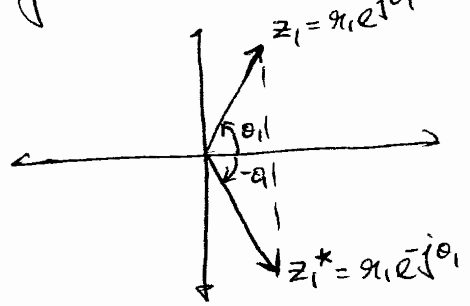
The complex conjugation of any complex number is done by reversing the sign of the imaginary part of the complex number.

In the polar form,

if $z = re^{j\theta} = r(\cos\theta + j\sin\theta)$

$z^* = re^{-j\theta} = r(\cos\theta - j\sin\theta)$

e.g: Geometric view



F] INVERSE

If $z = x+jy$, then the inverse of z is denoted by z^{-1} and is given by

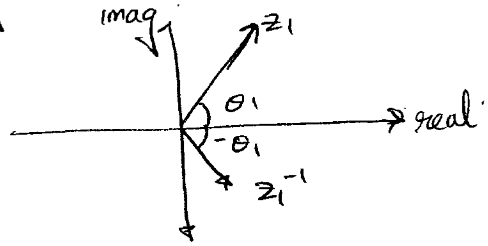
$$z^{-1} = \frac{1}{z} = \frac{1}{x+jy} = \frac{1}{x+jy}\cdot\frac{x-jy}{x-jy} = \frac{x-jy}{x^2+y^2}$$

$$= \underbrace{\frac{x}{x^2+y^2}}_{\text{real}} - j\underbrace{\frac{y}{x^2+y^2}}_{\text{imaginary}}$$

In polar form, if $z = re^{j\theta}$.

$$z^{-1} = \frac{1}{z} = \frac{1}{re^{j\theta}} = \frac{1}{r}e^{-j\theta}$$

e.g.: Geometrical view



III] POWERS AND ROOTS

A] DE MOIVRE'S FORMULA:

$$(\cos\theta + j\sin\theta)^N = \cos N\theta + j\sin N\theta$$

This can be proved using Euler's formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\therefore (e^{j\theta})^N = (\cos\theta + j\sin\theta)^N \quad - 1)$$

Now $(e^{j\theta})^N = e^{jN\theta} = \cos N\theta + j\sin N\theta \quad - 2)$.

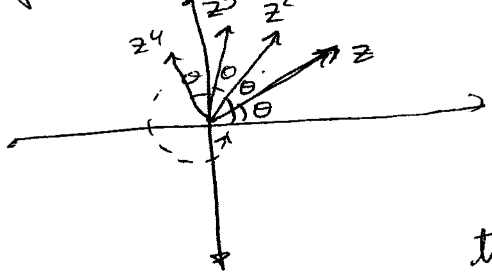
Equating 1) & 2), we get the expression for De-Moivre's Formula.

B] POWERS

The integer powers of a complex number can be defined as

$$z^N = (re^{j\theta})^N = r^N e^{jN\theta}$$

The geometric view is as shown below



Thus if $z = re^{j\theta}$

$z^2 = r^2 e^{j(2\theta)}$

$z^3 = r^3 e^{j(3\theta)}$

$\vdots$

if $r < 1$, then the magnitude of z keeps reducing as the power index increases.

c] ROOTS OF UNITY

Here, we find the solution to the equation

$z^N = 1$.

Now, $1 = e^{j2\pi l}$ where l is any integer

$\therefore\ z = e^{j\frac{2\pi l}{N}}$ $\quad l = 0 \cdots N-1$.

The above values of z are the N solutions to the equation $z^N = 1$.

D] MULTIPLE ROOTS

Suppose we have an equation of the form

$z^N = c$ where c is a complex constant

$c = |c| e^{j\phi}$.

We have to find all the values of z that satisfy the above equation.

We solve the equation using the following steps

A] express z in polar form

i.e. $LHS = z^N = (re^{j\theta})^N$

B] express the RHS as

$c = ce^{j2\pi l} = |c| e^{j\phi} e^{j2\pi l}$.

c] Equating A & B, we have

$r = |c|^{1/N}$

$e^{j\theta} = e^{j\left(\frac{\phi + 2\pi l}{N}\right)} \Rightarrow \theta = \frac{\phi}{N} + \frac{2\pi l}{N}$ $\quad l = 0, 1, \cdots N-1$

This provides us with the desired solution to the equation $z^N = c$