MAT 202

Introduction To Mathematics For DSP

Set 3: Z-Transform

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A transform is used to map a function from one domain (original domain) to another (transformed domain). The z-transform maps signals and systems from the time domain discussed earlier, into the z-domain. This kind of transformation is useful for the analysis of signals & systems, and offers insights which may not be apparent from the time-domain representation of signals.

From the geometric representation of the z-transform, one can easily decide whether the signal or system is stable or unstable. In case of systems, the z-transform helps us figure out the kind of filtering action being performed by the system. Thus, the z-transform is a useful mathematical tool for studying discrete-time signals & systems.

I] DEFINITION

The z-transform of a finite length signal $x[n]$ is given by

$$\mathcal{Z}\{x[n]\} = \sum_{n=0}^{N} x[n]\, z^{-n}$$

The term 'z' represents a complex variable

We can represent $\mathcal{Z}\{x[n]\}$ as $X(z)$

For an infinite length signal, the z-transform is given by

$$\mathcal{Z}\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

e.g: $x[n] = \{1, 0, 2, 3\}$

$$\therefore X(z) = \sum_{n=0}^{3} x[n]\, z^{-n}$$

$$= x[0] + x[1] z^{-1} + x[2] z^{-2} + x[3] z^{-3}$$

$$X(z) = 1 + 2z^{-2} + 3z^{-3}$$

e.g: $x[n] = \{\overset{-2}{4}, \overset{-1}{-3}, \underset{\uparrow}{\overset{0}{1}}, \overset{1}{2}, \overset{2}{5}\}$ $\leftarrow$ represents value at $n=0$

$$X(z) = \sum_{n=-2}^{2} x[n]\, z^{-n}$$

$$= x[-2] z^{2} + x[-1] z + x[0] + x[1] z^{-1} + x[2] z^{-2}$$

$$= 4z^{2} - 3z + 1 + 2z^{-1} + 5z^{-2}.$$

e.g: $x[n] = \{\ldots\ldots, 3, -1, \underset{\uparrow}{2}, 1, 3, \ldots\ldots\}$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-N}$$

$$= \ldots + x[-2]z^{+2} + x[-1]z^{+1} + x[0] + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} + \ldots$$

$$= \ldots + 3z^{2} - 1z + 2 + z^{-1} + 3z^{-2} + \ldots\ldots$$

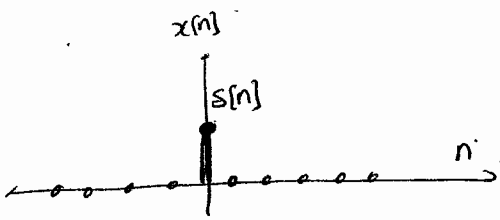
II] Z-TRANSFORM OF SOME COMMON FUNCTIONS :

A] DELTA FUNCTION (UNIT IMPULSE)

$$x[n] = \delta[n]$$

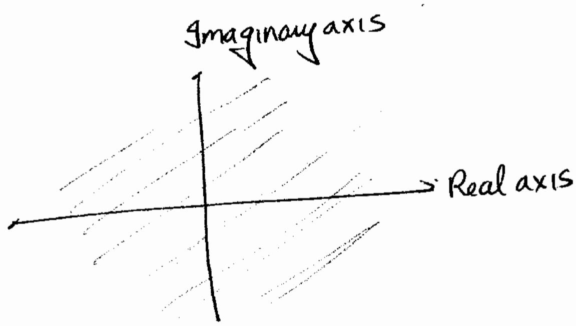
$$X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n}$$

$$\Rightarrow \boxed{X(z) = 1}$$



TIME DOMAIN

$\Longrightarrow$



Z- DOMAIN

Thus for a unit impulse, its z-transform has a finite constant value at every point 'z' in the z-plane.

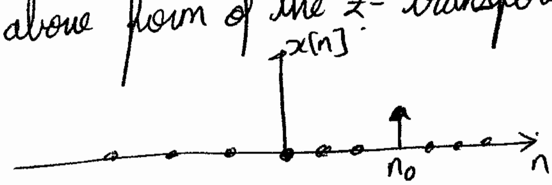
B] SHIFTED DELTA FUNCTION :

$$x[n] = \delta[n-n_0]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} \delta[n-n_0] z^{-n} = 0 + 0 + \cdots + 0 + z^{-n_0} + \ldots$$

$$= z^{-n_0}$$

Thus, since $\delta[n-n_0]$ has value '1' at $n=n_0$ and is '0' at all other 'n', we obtain the above form of the z-transform.



TIME DOMAIN

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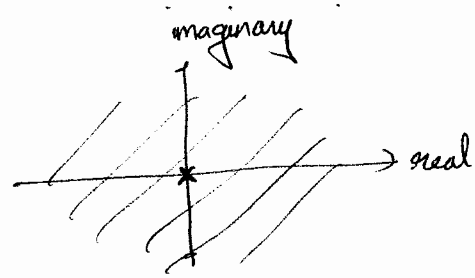


Z- DOMAIN :

We observe that $X(z)$ has a finite value at all points in the z-plane except $z=0$:

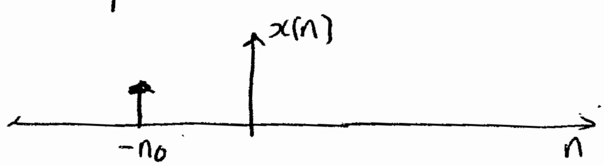
$$X(z) = z^{-n_0}$$
$$= \frac{1}{z^{n_0}}$$

$$\therefore X(0) = \frac{1}{0} = \infty \quad \{\text{for } n_0 > 0\}$$

We represent the value at point $z=0$ by a 'x' in the z-plane. This point is referred to as a 'POLE'.

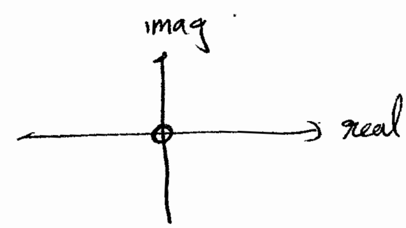
A pole is that point in the z-plane, at which the value of the z-transform is infinite. It is always denoted by a 'x' in the z-plane.

Now, if $n_0 > 0$, $\delta[n-n_0]$ is as shown below.



TIME DOMAIN

$\Longrightarrow$



z- DOMAIN

$$\therefore X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$
$$= 0 + 0 + \cdots + z^{n_0} + \cdots$$
$$= z^{n_0}$$

We observe that $X(z)$ has a finite non-zero value at every point except at $z=0$ where $X(z)=0$.

$$\therefore X(0) = (0)^{n_0} = 0$$

~~We prope~~ We represent the value of $X(z)$ at $z=0$ by a 'o' in the z-plane.

This point is referred to as a ~~zero~~ ZERO.

A zero is a point in the z-plane where the value of the z-transform $X(z)$ is simply zero. It is always indicated by a 'o' in the z-plane.

] UNIT STEP FUNCTION.

$$x[n] = u[n]$$

$$\therefore X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$
$$= \sum_{n=0}^{\infty} z^{-n}$$

$$\therefore X(z) = \frac{1}{1-z^{-1}} = \frac{1}{1-\frac{1}{z}} = \frac{1}{\frac{z-1}{z}}$$
$$= \frac{z}{z-1}$$

In the above calculation, we invoke the property of GEOMETRIC SERIES.

A geometric series is a series of the form

$1, a, a^2, a^3, a^4, a^5 \ldots$

In the above, every element = prev element × 'a'.

Thus, $\frac{a_{n+1}}{a_n} = \frac{a^{n+1}}{a^n} = a$ i.e ratio of $(n+1)^{th}$ element to the 'n^{th}' element is constant

We can express the sum of a geometric series as

$$\sum_{n=0}^{N} a^n = 1 + a + a^2 + \cdots + a^N = \frac{1-a^{N+1}}{1-a}$$

We derive this as follows:

Let $S_n = \sum_{n=0}^{N} a^n$

$$a \cdot S_n = \sum_{n=0}^{N} a^{n+1}$$

$$\therefore S_n - aS_n = \sum_{n=0}^{N} a^n - a\left(\sum_{n=0}^{N} a^n\right)$$
$$= (1 + a + a^2 + a^3 + \cdots + a^N) - a(1 + a + a^2 + \cdots + a^N)$$
$$= (1 + a + a^2 + \cdots + a^N) - (a + a^2 + a^3 + \cdots + a^{N+1})$$

$$\therefore (1-a)S_n = 1 - a^{N+1}$$

$$S_n = \frac{1-a^{N+1}}{1-a}$$

For $a<1$, as $N \to \infty$ (i.e length of the series increases to infinity)

$$a^{N+1} \to 0$$

$$\therefore \sum_{n=0}^{\infty} a^{N+1} = \frac{1-a^{N+1}}{1-a}$$

$$\Rightarrow \boxed{\sum_{n=0}^{\infty} a^{N+1} \approx \frac{1}{1-a}}$$ ← Useful property for z-transforms.

Coming back to the z-transform for the unit-step.

we have $X(z) = \frac{z}{z-1}$ {for $|z|>1$}

The condition, $|z|>1$ is important, since for $|z|<1$

$$X(z) = \sum_{n=0}^{\infty} z^{-n} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

Thus, if $|z|<1$

$\frac{1}{z} > 1$

hence $X(z) \to \infty$

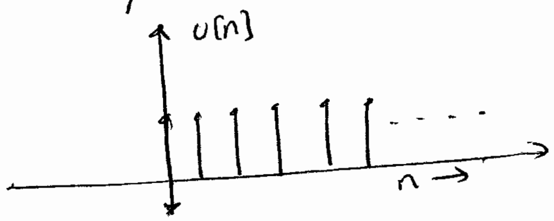
BUT if $|z|>1$

$\frac{1}{z} < 1$

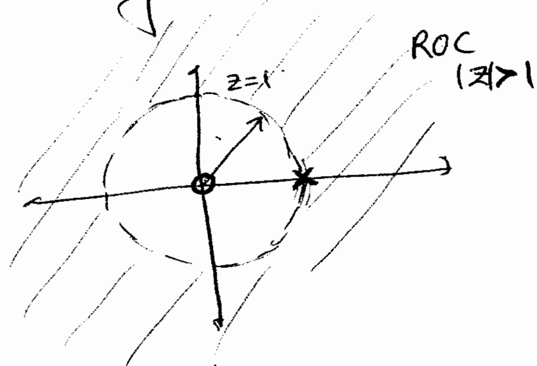
hence $X(z) = \frac{z}{z-1}$ {Using property of geometric series}

Thus, we observe that $X(z)$ converges to a finite value only if $z>1$. The points on the z-plane which satisfy the equation "$|z|>1$", together form the REGION OF CONVERGENCE (ROC) of the z-transform.

The REGION OF CONVERGENCE or ROC, specifies the region on the z-plane, where the z-transform has a finite value.



TIME DOMAIN



z-DOMAIN

The shaded region in the z-plane defines the region of convergence {$z>1$} for the z-transform of the unit-step function.

We also observe the following

$$X(z)\Big|_{z=0} = \frac{z}{z-1}\Big|_{z=0} = 0 \Rightarrow \text{zero at } z=0$$

$$X(z)\Big|_{z=1} = \frac{z}{z-1}\Big|_{z=1} = \infty \Rightarrow \text{pole at } z=1$$

The pole and zero are shown in the z-plane above.

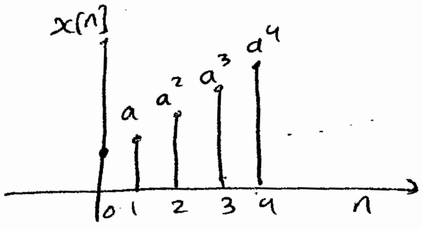
D] EXPONENTIAL FUNCTION

$$x[n] = a^n u[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}}$$

$$= \frac{z}{z-a} \qquad \text{ROC } \{|z|>|a|\}$$



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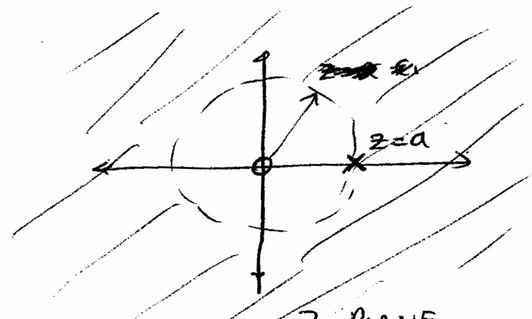


Z-PLANE

The z-plane has a pole at $z=a$ & a zero at $z=0$.

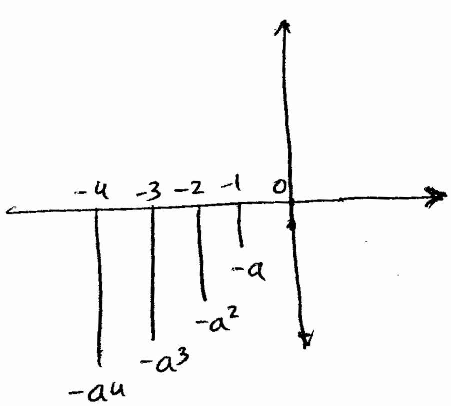
E] LEFT-SIDED EXPONENTIAL FUNCTION

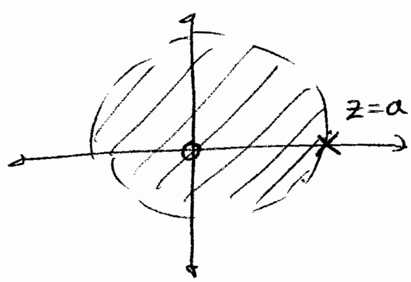
$$x[n] = -a^n u[-n-1]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \left[\sum_{n=0}^{\infty} a^{-n} z^n\right]$$

$$= 1 - \frac{1}{1-a^{-1}z} = \frac{z}{z-a} \qquad \text{ROC } \{|a^{-1}z|<1\} \Rightarrow |z|<|a|$$





Z- DOMAIN

TIME DOMAIN

We observe that the z-transform for the left-sided exponential function is the same as the z-transform for the right-sided exponential function discussed earlier. However, they differ in their ROCs, For the left sided exponential the ROC lies inside the circle $|z| = a$, while for the right-sided exponential, the ROC lies outside the circle $|z| = a$.

Thus the z-transform and the ROC uniquely define the z-transform domain representation of a time domain signal.

F] TWO SIDED EXPONENTIAL.

$$x[n] = a^n u[n] - b^n u[-n-1]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} - \sum_{n=-\infty}^{-1} b^n z^{-n}$$

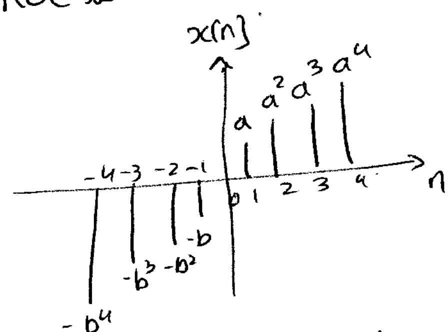
$$= \frac{1}{1-az^{-1}} + \frac{1}{1-bz^{-1}}$$

$$= \frac{2\left(1 - \frac{b+a}{2} z^{-1}\right)}{(1-az^{-1})(1-bz^{-1})}$$

The ROC of the first term is $|z| > |a|$ & ROC of the second term is $|z| < |b|$.

$\therefore$ if $|a| < |b|$, Then

ROC is: $|a| < |z| < |b|$



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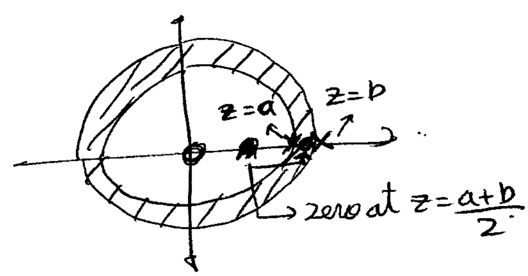


if $|a| > |b|$, Then ROC does not exist & the z-transform is not defined for the function.

g] FINITE LENGTH SEQUENCE.

$$x[n] = \begin{cases} a^n & 0 \le n < N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n}$$

$$= \frac{1-(az^{-1})^n}{1-az^{-1}}$$

$$= \frac{z^N - a^N}{z^{N-1}(z-a)}$$

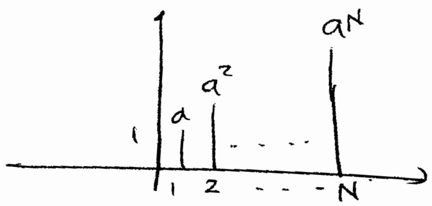
We can also express $X(z)$ as

$$X(z) = 1 + az^{-1} + a^2 z^{-2} + a^3 z^{-3} + \cdots + a^N z^{-N}$$

$$= 1 + \frac{a}{z} + \frac{a^2}{z^2} + \cdots + \frac{a^N}{z^N}$$

$$= \frac{z^N + az^{N-1} + a^2 z^{N-2} + \cdots + a^N}{z^N}$$

Thus, the ROC is the entire z-plane except $z = 0$.



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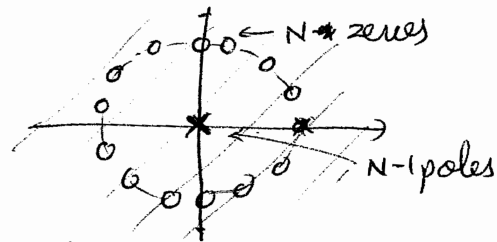

There are N zeros at $z = a e^{j\frac{2\pi k}{N}}$ and N-1 poles at the origin {based on the 1st equation}. There is also a pole at $z = a$.

The N-1 zeros locations are obtained as follows

$$X(z) = \frac{z^N - a^N}{z^{N-1}(z-a)}$$

zeros $\Rightarrow$ $z^N - a^N = 0$

$$z^N = a^N = a^N e^{j2\pi k}$$

$$\therefore z = (a^N e^{j2\pi k})^{1/N}$$

$$= a e^{j\frac{2\pi k}{N}} \qquad k = 0, 1, \cdots N-1$$

Thus all the zeros lie on the circle $|z| = |a|$, but at different angles, denoted by $\frac{2\pi k}{N}$.

III] Inversion Of The Z-Transform

The formal expression for the inverse z-transform is

$$x[n] = \frac{1}{2\pi j}\oint_c X(z)\, z^{n-1}\, dz.$$

There are three methods for evaluating the above complex integral

1) Direct evaluation, by contour integration
2) Power series expansion
3) Partial fraction expansion and table look-up.

The first method requires a solid background in analytic functions. So we will focus on 2) & 3) and come back to the direct evaluation if time permits.

A] Power Series Method

In this method, given the z-transform $X(z)$, we expand it in the form

$$X(z) = \sum_{n=-\infty}^{\infty} a_n z^{-n}$$

Then, by definition of z-transform, we obtain the time domain sequence $x[n]$ as

$x[n] = a_n$ for all n.

It is important to note that the series should be expressed in a form which will converge for the specified ROC.

e.g $X(z) = \dfrac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$

Obtain $x[n]$ for
a) ROC: $|z| > 1$
b) ROC: $|z| < 0.5$

a) Since ROC is in the exterior of a circle, $x[n]$ must be causal. Thus, we need a power series expansion in terms of the negative powers of z

$\therefore$ Performing long division

$$\begin{array}{r|l} & 1 + 1.5z^{-1} + 1.75z^{-2} \ldots \\ 1 - 1.5z^{-1} + 0.5z^{-2} & 1 \end{array}$$

$$- \quad 1 - 1.5z^{-1} + 0.5z^{-2}$$
$$\quad (+) \qquad (-)$$
$$1.5z^{-1} - 0.5z^{-2}$$
$$- \quad 1.5z^{-1} \underset{(+)}{-} 2.25z^{-2} \underset{(-)}{+} 0.75z^{-3}$$
$$1.75z^{-2} - 0.75z^{-3}$$

$$\therefore X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}} = 1 + 1.5z^{-1} + 1.75z^{-2} + \ldots$$

$$\therefore x[n] = \{\underset{\uparrow}{1}, 1.5, 1.75, \ldots\}$$

b) ROC : $|z| < 0.5$

In this case ROC is inside a circle $\Rightarrow x[n]$ is anticausal

We need to obtain the power series in terms of positive powers of z.

This requires a minor alteration in the way we perform long-division

$$\therefore X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}} = \frac{1}{\frac{1}{2}z^{-2} - 1.5z^{-1} + 1}$$

$$\begin{array}{r|l} & 2z^2 + 6z^3 + 14z^4 \ldots \\ \frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & 1 \end{array}$$

$$- \quad 1 - 3z + 2z^2$$
$$\quad (+) \qquad (-)$$
$$3z - 2z^2$$
$$(-)3z \underset{(+)}{-} 9z^2 \underset{(-)}{+} 6z^3$$
$$7z^2 - 6z^3 \ldots$$

$$\therefore X(z) = 2z^2 + 6z^3 + 14z^4 + \ldots$$

$$x[n] = \{\ldots, 14, 6, 2, 0, \underset{\uparrow}{0}\}$$

Thus, we observe that for the same z-transform, difference in ROC results in completely different signals.

The Power Series method does not provide a closed form solution, and is used only if one has to obtain the first few samples of the signal.

B] PARTIAL FRACTION EXPANSION

If $X(z)$ be a rational function which can be expressed as

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

then if $M < N$, $X(z)$ is said to be a proper rational function & we can use the partial fraction expansion method.

If $M > N$, $X(z)$ is said to be an improper rational function. This can be expressed as the sum of a polynomial & a rational function. We can then perform a partial fraction expansion of the rational part.

e.g: improper rational function

$$X(z) = \frac{1 + 2z^{-1} + 3z^{-2} + 4z^{-3}}{2 + 3z^{-1} + 4z^{-2}}$$

$$= \frac{4z^{-3} + 3z^{-2} + 2z^{-1} + 1}{4z^{-2} + 3z^{-1} + 1}$$

$$= \underbrace{z^{-1}}_{\text{polynomial}} + \underbrace{\frac{1}{4z^{-2} + 3z^{-1} + 1}}_{\text{proper rational } f^{n}}$$

In case of the partial fraction expansion, we consider 2 cases

a) Distinct poles

Here, we try to express $X(z)$ in the form

$$\frac{X(z)}{z} = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \cdots + \frac{A_n}{z - p_n}$$

We compare the above with the original expressn for $X(z)$

$$A_1 = \frac{X(z)}{z}(z - p_1)\Big|_{z = p_1}$$

$$A_2 = \frac{X(z)}{z}(z - p_2)\Big|_{z = p_2}$$

$$\vdots$$

Having obtained the values of $A_1, A_2 \cdots A_n$ and knowing the ROC, we can write ~~we can express x(z) as~~ an expression for $x[n]$.

e.g: $X(z) = \dfrac{z}{z^2 - 3z + 2}$

$$\therefore \frac{X(z)}{z} = \frac{1}{z^2 - 3z + 2} = \frac{1}{(z-3)(z-2)} = \frac{A}{z-3} + \frac{B}{z-2}$$

$$A = \frac{X(z)}{z}(z-3)\Big|_{z=3} = \frac{1}{(z-2)}\Big|_{z=3} = \frac{1}{3-2} = 1$$

$$B = \frac{X(z)}{z}(z-2)\Big|_{z=2} = \frac{1}{z-3}\Big|_{z=2} = \frac{1}{2-3} = -1$$

$$\therefore \frac{X(z)}{z} = \frac{1}{z-3} - \frac{1}{z-2}$$

$$X(z) = \frac{z}{z-3} - \frac{z}{z-2}$$

$$= \frac{1}{1-3z^{-1}} - \frac{1}{1-2z^{-1}}$$

If ROC: $|z| > 3$

$$x[n] = (3)^n u[n] - (2)^n u[n]$$

If ROC: $2 < |z| < 3$

$$x[n] = -(3)^n u[-n-1] + (2)^n u[n]$$

If ROC: $|z| < 2$

$$x[n] = -(3)^n u[-n-1] + (2)^n u[n]$$

b) Multiple Order poles:

$$\frac{X(z)}{z} = \frac{A_1}{z-p_1} + \frac{A_2}{(z-p_2)} + \frac{A_n}{(z-p_n)} + \frac{A_{n+1}}{(z-p_n)^2}$$

e.g: $X(z) = \dfrac{1}{(1+z^{-1})(1-z^{-1})^2}$

$$\frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2} = \frac{A}{(z+1)} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$A = \frac{X(z)}{z}(z+1)\Big|_{z=-1} = \frac{z^2}{(z-1)^2} = \frac{1}{4}$$

$$C = \frac{X(z)}{z}(z-1)^2\Big|_{z=1} = \frac{z^2}{z+1}\Big|_{z=1} = \frac{1}{2}$$

$$B = \frac{d}{dz}\left[(z-1)^2\frac{X(z)}{z}\right]\Big|_{z=1} = \frac{d}{dz}\left[\frac{z^2}{z+1}\right]\Big|_{z=1} = \frac{(z+1)(2z)-z^2}{(z+1)^2}\Big|_{z=1} = \frac{4-1}{4} = \frac{3}{4}$$

$$\left\{ \because \frac{X(z)(z-1)^2}{z} = \frac{(z-1)^2 A(z)}{z+1} + \frac{(z-1)}{2}B + C \right.$$

$$\Rightarrow \frac{d}{dz}\left[\frac{X(z)}{z}(z-1)^2\right] = 2(z-1)\frac{d}{dz}\frac{(A(z))}{z+1} + B$$

$$\left. \Rightarrow \frac{d}{dz}\left[\frac{X(z)}{z}(z-1)^2\right]\Big|_{z=0} = B \right\}$$

$$\therefore X(z) = \frac{1/4\, z}{(z+1)} + \frac{3/4\, z}{(z-1)} + \frac{1/2\, z}{(z-1)^2}$$

$$\therefore x[n] = \frac{1}{4}(-1)^n u[n] + \frac{3}{4}u[n] + \frac{1}{2}n u[n]$$