MAT 202

Introduction To Mathematics For DSP

Set 4: Convolutions & Correlations

Convolution is used to obtain the output of a system, given the input signal & the impulse response of the system. Correlation is used to compare signals and determine signal characteristics such as pitch and delay. We consider these in more detail

I] CONVOLUTION:

Convolution is applicable to systems that are LINEAR and TIME INVARIANT (LTI)

Let $x[n]$ be the input to a LTI system with impulse response $h[n]$ and let $y[n]$ be the output

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

Then, $y[n]$ can be expressed as the convolution of the input sequence $x[n]$ and the system response $h[n]$

$$\therefore\ y[n] = x[n] * h[n] \qquad \leftarrow \{\text{Notation}\}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

Two ways of expressing the convolution sum.

We demonstrate 2 different ways of deriving the above.

A] METHOD 1:

The input sequence $x[n]$ can be expressed as a sum of ~~imp~~ time delayed impulses as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

This is known as the SIFTING PROPERTY.

The output $y[n]$ can be expressed as

$$y[n] = H\{x[n]\} \qquad \text{where } H \rightarrow \text{transformation by the system}$$

$$= H\left\{\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]\right\}$$

$$= \sum_{k=-\infty}^{\infty} H\{x[k]\,\delta[n-k]\} \qquad \rightarrow \{\text{invoking superposition property}\}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\,H\{\delta[n-k]\}$$

$$\therefore y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

B] <u>METHOD 2</u> :-

$$\delta[n] \longrightarrow \boxed{h[n]} \longrightarrow h[n] \quad \rightarrow \text{ Sifting property}$$

$$\delta[n-k] \longrightarrow \boxed{h[n]} \longrightarrow h[n-k] \quad \rightarrow \text{ Time invariance}$$

$$x[k]\delta[n-k] \longrightarrow \boxed{h[n]} \longrightarrow x[k]h[n-k] \quad \rightarrow \text{ scalar multiplication}$$

$$\underbrace{\sum_{k=0}^{\infty} x[k]\delta[n-k]}_{\Downarrow \; x[n]} \longrightarrow \boxed{h[n]} \longrightarrow \underbrace{\sum_{k=0}^{\infty} x[k]h[n-k]}_{\Downarrow \; y[n]} \quad \rightarrow \text{ linearity}$$

e.g] $x[n] = [\underset{\uparrow}{1}\ 2\ 1\ 3]$

$h[n] = [\underset{\uparrow}{1}\ 3\ 1]$

$\therefore y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

$y[0] = x[0]h[0] = 1$

$y[1] = x[0]h[1] + x[1]h[0] = (1)(3) + (2)(1) = 5$

$y[2] = x[0]h[2] + x[1]h[1] + x[2]h[0]$

$\quad = (1)(1) + (2)(3) + (1)(1) = 8$

$y[3] = x[0]h[3] + x[1]h[2] + x[2]h[1] + x[3]h[0]$

$\quad = 0 + (2)(1) + (1)(3) + (3)(1) = 8$

$y[4] = x[2]h[2] + x[3]h[1] = (1)(1) + (3)(3) = 10$

$y[5] = x[3]h[2] = (3)(1) = 3$

$\therefore y = \{\underset{\uparrow}{1}\ 5\ 8\ 8\ 10\ 3\}$

<u>NOTE</u>: If $x[n]$ is of length M, and $h[n]$ is of length N

Length of $y[n]$ is $M+N-1$.

4] PROPERTIES OF CONVOLUTION :

1) COMMUTATIVE :

$$x_1[n] * x_2[n] = x_2[n] * x_1[n].$$

2) ASSOCIATIVE :

$$(x_1[n] * x_2[n]) * x_3[n] = x_1[n] * (x_2[n] * x_3[n])$$

3) DISTRIBUTIVE :

$$x_1[n] * (x_2[n] + x_3[n]) = x_1[n] * x_2[n] + x_1[n] * x_3[n].$$

II] CORRELATION

A measure of similarity between two energy (finite length) signals $x[n]$ and $y[n]$ is given by the CROSS CORRELATION

$$r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]\, y[n-l].$$

'l' is the 'lag' or 'time shift' between the 2 signals.

The ordering 'xy' indicates that $x[n]$ is the reference signal which is fixed, while the sequence $y[n]$ is shifted with respect to $x[n]$.

If we wish to use $y[n]$ as the reference & shift $x[n]$ with respect to $y[n]$, then the cross-correlation is given by

$$r_{yx}[l] = \sum_{n=-\infty}^{\infty} y[n]\, x[n-l]$$

$$= \sum_{m=-\infty}^{\infty} y[m+l]\, x[m]$$

$$= r_{xy}[-l].$$

AUTOCORRELATION of a sequence $x[n]$ is given by

$$r_{xx}[l] = \sum_{n=-\infty}^{\infty} x[n]\, x[n-l].$$

A] PROPERTIES OF AUTOCORRELATION & CROSS CORRELATION

Let $x[n]$ and $y[n]$ be two energy signals

$$\therefore \sum_{n=-\infty}^{\infty} (a x[n] + y[n-l])^2 = a^2 \sum_{n=-\infty}^{\infty} x^2[n] + 2a \sum_{n=-\infty}^{\infty} x[n] y[n-l] + \sum_{n=-\infty}^{\infty} y^2[n-l]$$

$$= a^2 r_{xx}(0) + 2a\, r_{xy}(l) + r_{yy}(0) \geq 0$$

Now, $a^2 r_{xx}[0] + 2a\, r_{xy}[l] + r_{yy}[0] \geq 0$

$$\Rightarrow \begin{bmatrix} a & 1 \end{bmatrix} \begin{bmatrix} r_{xx}(0) & r_{xy}(l) \\ r_{xy}(l) & r_{yy}(0) \end{bmatrix} \begin{bmatrix} a \\ 1 \end{bmatrix} \geq 0$$

$$\Rightarrow \begin{bmatrix} r_{xx}(0) & r_{xy}(0) \\ r_{xy}(0) & r_{yy}(0) \end{bmatrix} \geq 0$$

$$\Rightarrow r_{xx}(0)\, r_{yy}(0) - r_{xy}^2(l) \geq 0$$

$$\Rightarrow r_{xy}(l) \leq \sqrt{r_{xx}(0)\, r_{yy}(0)}$$

Now $r_{xx}(0) = \sum_{n=-\infty}^{\infty} x[n]^2 = E_x$

$r_{yy}(0) = \sum_{n=-\infty}^{\infty} y[n]^2 = E_y$

$$\therefore \boxed{r_{xy}(l) \leq \sqrt{E_x E_y}}$$

If we set $x[n] = y[n]$, then

$\boxed{r_{xx}[l] \leq r_{xx}(0)}$ i.e $r_{xx}[l] \leq E_x$

B] NORMALIZED CORRELATIONS

The normalized correlation is expressed as

$$\rho_{xy}[l] = \frac{r_{xy}[l]}{\sqrt{r_{xx}[0]\, r_{yy}[0]}} \qquad |\rho_{xy}[l]| \leq 1$$

$$\rho_{xx}[l] = \frac{r_{xx}[l]}{r_{xx}[0]} \qquad |\rho_{xx}[l]| \leq 1$$

C] CORRELATION OF POWER SIGNALS:

For power signals, the cross-correlation sequence is defined as.

$$r_{xy}[l] = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=-K}^{K} x[n]y[n-l]$$

& the autocorrelation is defined as

$$r_{xx}[l] = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=-K}^{K} x[n]x[n-l]$$

In case of periodic signals, the autocorrelation & cross-correlation are given by

$$r_{xy}[l] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]y[n-l] \qquad r_{xx}[l] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]x[n-l],$$

where 'N' is the fundamental period of the signals $x[n]$ & $y[n]$.

D] EXAMPLE:

Let $x[n] = \{1, 0, 2, 3\}$

$y[n] = \{2, 3\}$

$\therefore r_{xy}[0] = x[0]y[0] = 2$

$r_{xy}[1] = x[1]y[0] + x[2]y[1] = 6$

$r_{xy}[2] = x[2]y[0] + x[3]y[1] = 13$

$r_{xy}[3] = x[3]y[0] = (3)(2) = 6$

Thus, at a lag of 2, there is maximum correlation between $x[n]$ & $y[n]$

E] FINDING THE PERIOD OF A PERIODIC SIGNAL CORRUPTED BY NOISE:

Let $x[n]$ be a periodic signal corrupted by random noise $d[n]$ resulting in the signal $w[n] = x[n] + d[n]$.

Let N be the period of signal $x[n]$

We observe the signal $w[n]$ for time $M >> N$.

$$r_{ww}[l] = \frac{1}{M} \sum_{n=0}^{M-1} w[n]w[n-l]$$

$$= \frac{1}{M} \sum_{n=0}^{M-1} (x[n] + d[n])(x[n-l] + d[n-l]).$$

$$\therefore r_{ww}[l] = \frac{1}{M}\sum_{n=0}^{M-1} x[n]\,x[n-l] + \frac{1}{M}\sum_{n=0}^{M-1} x[n]\,d[n-l] + \frac{1}{M}\sum_{n=0}^{M-1} x[n-l]\,d[n] + \frac{1}{M}\sum_{n=0}^{M-1} d[n]\,d[n-l]$$

$$= r_{xx}[l] + r_{xd}[l] + r_{dx}[l] + r_{dd}[l]$$

$r_{xx}[l]$ will have peaks at $l = 0, N, 2N \cdots$

Since $x[n]$ & $d[n]$ are uncorrelated, $r_{xd}[l]$ & $r_{dx}[l]$ will have small values.

$r_{dd}[l]$ has a peak at $l=0$ & decreases beyond that time.

Thus, ~~auto~~correlation can be used to obtain the period of a periodic sequence.