MAT 202.

Introduction To Mathematics For DSP.

Set 5: Discrete Time Fourier Transform.

The FOURIER TRANSFORM is used to transform a signal from the time domain into the frequency domain. The DISCRETE TIME FOURIER TRANSFORM (DTFT) or simply the Fourier transform maps the time domain sequence into a continuous function of the frequency. It is an alternative representation of the signal $x[n]$ in terms of the complex exponential sequence $\{e^{-j\omega n}\}$, where 'ω' is a real frequency variable.

I] DEFINITION :

The discrete time Fourier transform $X(e^{j\omega})$ of a sequence $x[n]$ is defined by:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$X(e^{j\omega})$ is a complex function of 'ω' and can be written as:

$$X(e^{j\omega}) = X_{re}(e^{j\omega}) + j X_{im}(e^{j\omega}) \quad \longrightarrow \text{RECTANGULAR FORM}$$

$$X(e^{j\omega}) = |X(e^{j\omega})| e^{j\theta(\omega)} \quad \longrightarrow \text{POLAR FORM}$$

where $|X(e^{j\omega})|$ is called the MAGNITUDE FUNCTION

& $\theta(\omega) = \arg\{X(e^{j\omega})\}$ is called the PHASE FUNCTION:

e.g: $x[n] = \alpha^n u[n]$ $\quad |\alpha| < 1$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n$$

$$= \frac{1}{1-\alpha e^{-j\omega}}$$

$X(e^{j\omega})$ is continuous and periodic in 'ω', with a period of 2π.

$$\therefore X(e^{j(\omega+2\pi k)}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j(\omega+2\pi k)} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega} \cdot e^{-j2\pi k}$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega}$$

$$= X(e^{j\omega})$$

The INVERSE FOURIER TRANSFORM is defined as:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

This can be proved as follows:

$$LHS = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\,e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{m=-\infty}^{\infty} x[m]\,e^{-j\omega m}\cdot e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} x[m]\int_{-\pi}^{\pi} e^{+j\omega(n-m)}\,d\omega$$

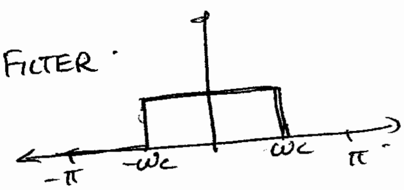
$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} x[m]\,\frac{2\sin((n-m)\pi)}{n-m}$$

$$= x[n]$$

This follows from the fact that

$$\frac{\sin \pi n}{\pi n} = 1 \quad \text{if } n = 0$$
$$= 0 \quad \text{otherwise}$$

LOW PASS FILTER



e.g: Let $H_{LP}(e^{j\omega}) = \begin{cases} 1 & 0 \le |\omega| \le \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$

$\therefore$ The inverse DFT is given by:

$$h_{LP}[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\,d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn}\right]$$

$$= \frac{\sin(\omega_c n)}{\pi n} \qquad -\infty < n < \infty$$

II] CONVERGENCE CONDITION:

If $x[n]$ is ABSOLUTELY SUMMABLE i.e $\sum_{n=-\infty}^{\infty}|x[n]| < \infty$, then

$$|X(e^{j\omega})| = \left|\sum_{n=-\infty}^{\infty} x[n]\,e^{-j\omega n}\right| \le \sum_{n=-\infty}^{\infty}|x[n]| < \infty$$

Thus, if a sequence is absolutely summable, it is a sufficient condition for the Fourier Transform to exist.

e.g: $x[n] = u[n]$.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=0}^{\infty} e^{-j\omega n}$$

Since $\sum_{n=0}^{\infty} |e^{-j\omega n}| \to \infty$, the DTFT does not exist.

Now, $\sum_{n=-\infty}^{\infty} |x[n]|^2 \leq \left(\sum_{n=-\infty}^{\infty} x[n] \right)^2$.

Hence an absolutely summable sequence has finite energy. But a finite energy sequence may not be absolutely summable.

For such signals, we relax the condition of ~~absolute conv~~ uniform convergence to MEAN SQUARE CONVERGENCE.

$$\lim_{K \to \infty} \int_{-\pi}^{\pi} |X(e^{j\omega}) - X_K(e^{j\omega})|^2 d\omega = 0$$

This means that ~~even~~ if the energy in the error $|X(e^{j\omega}) - X_K(e^{j\omega})|$ tends towards zero, but the actual error ~~ex~~ $|X(e^{j\omega}) - X_K(e^{j\omega})|$ does not tend to zero, the Fourier transform exists {e.g: $\sin(\omega_c n)/\pi n$}

III] Relationship Of The Fourier Transform To The Z-Transform:

We have seen that the z-transform of a sequence is given by:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}. \qquad R.O.C: r_2 < |z| < r_1$$

We can express z in polar form as $z = re^{j\omega}$

$\therefore$ Within the ROC, the z-transform can be written as

$$X(z)\Big|_{z=re^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n]\, r^{-n} e^{-j\omega n}$$

If $X(z)$ converges at $|z|=1$, then

$$X(z)\Big|_{z=e^{j\omega}} = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Thus, the Fourier transform can be viewed as the z-Transform evaluated on the unit circle. If $X(z)$ does not converge in the region $|z|=1$ i.e if the unit circle is not contained in the region of convergence of $X(z)$, the Fourier transform does not exist.

IV] PROPERTIES OF DTFT :

1) LINEARITY :

$$x_1[n] \xleftrightarrow{DTFT} X_1(e^{j\omega}) \quad \text{and} \quad x_2[n] \xleftrightarrow{DTFT} X_2(e^{j\omega})$$

then,

$$a x_1[n] + b x_2[n] \xleftrightarrow{DTFT} a X_1(e^{j\omega}) + b X_2(e^{j\omega})$$

2) TIME SHIFTING :

$$x[n-n_0] \xleftrightarrow{DTFT} e^{-j\omega n_0} X(e^{j\omega})$$

Proof : $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$

$$\therefore x[n-n_0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega(n-n_0)} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} [X(e^{j\omega}) e^{-j\omega n_0}] e^{j\omega n} d\omega$$

$$\therefore F\{x[n-n_0]\} = X(e^{j\omega}) e^{-j\omega n_0}$$

3) MODULATION :

$$e^{j\omega_0 n} x[n] \xleftrightarrow{DTFT} X(e^{j(\omega-\omega_0)})$$

Proof : $X(e^{j(\omega-\omega_0)}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega-\omega_0)n}$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{j\omega_0 n} e^{-j\omega n}$$

$$= F\{x[n] e^{j\omega_0 n}\}$$

4) DIFFERENTIATION IN FREQUENCY:

$$n x[n] \xleftrightarrow{DTFT} j \frac{dX(e^{j\omega})}{d\omega}$$

Proof: $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} \frac{d}{d\omega} \left(x[n] e^{-j\omega n} \right)$$

$$= -\sum_{n=-\infty}^{\infty} j n x[n] e^{-j\omega n}$$

$$\therefore j \frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} n x[n] e^{-j\omega n}$$

$$= F\{n x[n]\}$$

5) TIME REVERSAL:

$$x[-n] \xleftrightarrow{DTFT} X(e^{-j\omega})$$

Proof: $$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$\therefore x[-n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{-j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{-j\omega}) e^{j\omega n} d\omega$$

6) CONJUGATION:

$$x^*[n] \xleftrightarrow{DTFT} X^*(e^{-j\omega})$$

Proof: $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$X^*(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x^*[n] e^{j\omega n}$$

$$X^*(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

7) CONVOLUTION OF SIGNALS:

$$x_1[n] * x_2[n] \xleftrightarrow{DTFT} X_1(e^{j\omega}) \cdot X_2(e^{j\omega}).$$

Proof: $F\{x_1[n] * x_2[n]\} = \sum_{n=-\infty}^{\infty} (x_1[n] * x_2[n])\, e^{-j\omega n}$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k]\, e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k]\, e^{-j\omega(n-k)}\, e^{-j\omega k}$$

$$= \sum_{k=-\infty}^{\infty} x_1[k]\, e^{-j\omega k} \sum_{n=-\infty}^{\infty} x_2[n-k]\, e^{-j\omega(n-k)}$$

$$= X_1(e^{j\omega}) \cdot X_2(e^{j\omega}).$$

$\Rightarrow$ $\therefore$ Convolution in time domain = Multiplication in frequency domain.

8) MULTIPLICATION OF SIGNALS:

$$x_1[n] * x_2[n] \xleftrightarrow{DTFT} \frac{1}{2\pi} X_1(e^{j\omega}) * X_2(e^{j\omega}).$$

Proof: $F\{x_1[n] \cdot x_2[n]\} = \sum_{n=-\infty}^{\infty} x_1[n] \cdot x_2[n]\, e^{-j\omega n}$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{j\theta}) \cdot e^{j\theta n}\, d\theta \cdot x_2[n]\, e^{-j\omega n}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{j\theta}) \cdot \sum_{n=-\infty}^{\infty} x_2[n]\, e^{-j(\omega-\theta)n}\, d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{j\theta}) \cdot X_2(e^{j(\omega-\theta)})\, d\theta$$

$$= \frac{1}{2\pi} X_1(e^{j\omega}) * X_2(e^{j\omega}).$$

Multiplication in time domain = convolution in frequency domain.

PARSEVAL'S RELATION:

Signal energy $= \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2\, d\omega.$

General case: $\sum_{n=-\infty}^{\infty} x_1[n]\, x_2^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{j\omega})\, X_2^*(e^{j\omega})\, d\omega.$

Proof:- $\sum_{n=-\infty}^{\infty} x_1[n]\, x_2^*[n] = \sum_{n=-\infty}^{\infty} x_1[n] \frac{1}{2\pi}\int_{-\pi}^{\pi} X_2^*(e^{j\omega})\cdot e^{-j\omega n}\, d\omega$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_2^*(e^{j\omega}) \sum_{n=-\infty}^{\infty} x_1[n]\, e^{-j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\omega})\, X_2^*(e^{j\omega})\, d\omega$$

Substituting $x_2^*[n]$ by $x_1^*[n]$, we have

$$\sum_{n=-\infty}^{\infty} x_1[n]\, x_1^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\omega})\, X_1^*(e^{j\omega})\, d\omega$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} |x_1[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X_1(e^{j\omega})|^2\, d\omega \rightarrow \text{Parseval's Theorem.}$$

V] LIMITATIONS OF DTFT

Even though the DTFT is a useful tool for analyzing signals in the frequency domain, it is not used in practical data analysis for the following reasons.

1) We assume that the signal $x[n]$ is known at all times i.e $-\infty < n < \infty$. However, in practice we do not have an infinite amount of data, and must work with finite data.

2) $X(e^{j\omega})$ is a continuous function of the frequency 'ω'. Hence, we cannot use a digital computer for computing or processing a continuum (infinite number) of values.

These limitations lead us to the discrete Fourier transform (DFT) and Fast Fourier transform (FFT) which can be practically implemented in digital signal processors.