MAT 202

Introduction To Mathematics For DSP

Set 7: Discrete Fourier Transform (DFT)

– Niranjan Shetty

The DTFT, as we have seen earlier, is continuous in frequency ~~thus~~ and hence cannot be used efficiently. However, it is seen that for a length N sequence, ~~the~~ only N values of the DTFT $X(e^{j\omega})$ are needed to determine $x[n]$ and hence $X(e^{j\omega})$ uniquely. This leads to the concept of the Discrete Fourier Transform, which has made DSPs physically realizable.

I] DEFINITION:

The DFT of a finite length sequence can be obtained by taking samples of its DTFT $X(e^{j\omega})$ at $\omega_k = \frac{2\pi k}{N}$, where N is the length of $x[n]$.

$$\therefore \text{DFT}\{x[n]\} = X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi nk}{N}} = X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}} \qquad 0 \le k \le N-1$$

We represent $e^{-j\frac{2\pi}{N}}$ as W_N.

$$\therefore X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad 0 \le k \le N-1$$

The INVERSE DISCRETE FOURIER TRANSFORM (IDFT) is given by

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \qquad 0 \le n \le N-1$$

<u>Derivation</u>:-

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

$$\therefore \sum_{k=0}^{N-1} X[k] W_N^{-kl} = \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} x[n] W_N^{(n-l)k}$$

$$= \sum_{n=0}^{N-1} x[n] \sum_{k=0}^{N-1} W_N^{(n-l)k}$$

Now $\sum_{k=0}^{N-1} W_N^{(n-l)k} = N$ for $n - l = rN$

$= 0$ otherwise

$$\therefore \sum_{k=0}^{N-1} X[k] W_N^{-kl} = N x[l]$$

$$\Rightarrow x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \qquad \{\text{changing variables}\}$$

e.g: Let $x[n] = \{x[0]\ x[1]\ x[2]\ x[3]\}$

$$\therefore X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk}$$

$$X[0] = x[0] + x[1]W_N^0 + x[2]W_N^0 + x[3]W_N^0$$

$$= x[0] + x[1] + x[2] + x[3]$$

$$X[1] = x[0] + x[1]W_N^{1} + x[2]W_N^{2} + x[3]W_N^3$$

$$X[2] = x[0] + x[1]W_N^2 + x[2]W_N^4 + x[3]W_N^6$$

$$X[3] = x[0] + x[1]W_N^3 + x[2]W_N^6 + x[3]W_N^9$$

This can be written in matrix form as

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_N^1 & W_N^2 & W_N^3 \\ 1 & W_N^2 & W_N^4 & W_N^6 \\ 1 & W_N^3 & W_N^6 & W_N^9 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

$$X = D_N \mathbf{x}$$

where $X = [X[0]\ X[1]\ X[2]\ X[3]]^T$

$\mathbf{x} = [x[0]\ x[1]\ x[2]\ x[3]]$

$$D_N = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_N^1 & W_N^2 & W_N^3 \\ 1 & W_N^2 & W_N^4 & W_N^6 \\ 1 & W_N^3 & W_N^6 & W_N^9 \end{bmatrix}$$

Now $e^{j\frac{2\pi nk}{N}} = e^{j\frac{2\pi nk}{N}} \cdot e^{j2\pi n}$

$$= e^{j\frac{2\pi n(k+N)}{N}}$$

$$\Rightarrow W_N^{nk} = W_N^{nk+N}$$

$\therefore$ In our case $N = 4$

$\therefore W_4^4 = 1$

$W_4^6 = W_4^{2+4} = W_4^2$

$W_4^9 = W_4^{1+8} = W_4^1$

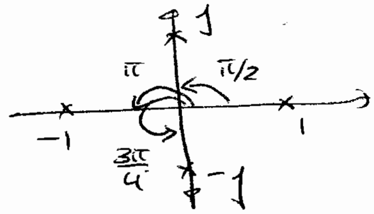
$\therefore$ D_4 can be written as

$$D_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & 1 & W_4^2 \\ 1 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix}.$$

Further, $W_4^1 = e^{j\frac{2\pi}{4}} = e^{j\frac{\pi}{2}} = j$

$W_4^2 = e^{j\frac{4\pi}{4}} = e^{j\pi} = -1$

$W_4^3 = e^{j\frac{3\pi}{2}} = -e^{j\frac{\pi}{2}} = -j$.



$$\therefore D_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}.$$

For obtaining the inverse DFT (IDFT), the matrix representation is given by:

$$x = \frac{1}{N} D_N^* X$$

$\therefore$ In case of the 4-point DFT

$$x = \frac{1}{4} D_4^* X$$

$$D_4^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}.$$

e.g: Let $x[n] = \{1 \quad 2 \quad 1 \quad 2\}$ $\qquad \{X = D_4 x\}$

$$\therefore X[k] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}$$

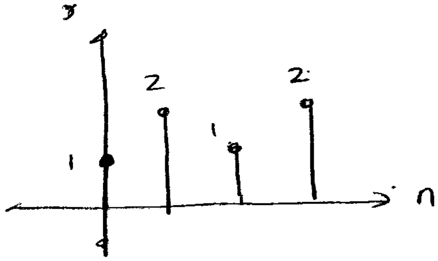
To obtain $x[n]$, we take the inverse DFT

$\cancel{x[n] = \frac{1}{4} D_4^* X[k]}$

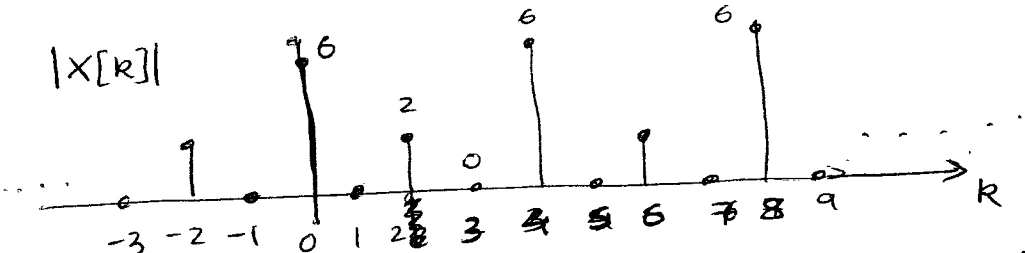
$\therefore \quad x = \frac{1}{N} D_N^* X$

$$\therefore \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

$\therefore \quad x[n]$



$|X[k]|$



We observe that DFT of $x(n)$ is a periodic sequence with period N. However, only N values of the DFT $X(k)$ are necessary for reconstructing $x(n)$.