Frequency Synchronization and Multi-stability in Power Grids

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SJ and Francesco Bullo. Synchronization of Kuramoto Oscillators via Cutset Projections. IEEE Transactions on Automatic Control, 2019.

SJ and Elizabeth Y. Huang and Kevin D. Smith and Francesco Bullo. Flow and Elastic Networks on the n-torus: Geometry, Analysis, and Computation. SIAM Review, accepted, 2021.

Introduction: Large-scale Nonlinear Networks

Power grids

Brain neural network

Transportation network

Nonlinearity:

- Multiple equilibria
- Transient stability
- Cluster synchronization

Large-scale:

- Stochastic
- Distributed

- *"... As [power] systems become more heavily loaded, nonlinearities play an increasingly important role in power system behavior ... "* [I. Hiskens,1995]
- *"... in Oahu, Hawaii, at least 800,000 micro-inverters interconnect photovoltaic panels to the grid... "* [IEEE Spectrum, 2015]

Presentation outline

Frequency Synchronization and Multi-stability in Power Grids

- oscillator networks and power grids
- transition from frequency synchronization to incoherency
  - cutset projection
  - sufficient synchronization conditions
- multiplicity of equilibria
  - winding cells
  - contractive iterations for computing equilibria

Phenomenon #1: Transition form sync to incoherency

Frequency synchronization:

- Frequency synchronization is crucial for functionality and operation of power grids.
- Power electronic devices are designed to work at 60 *Hz* in the US (50 *Hz* in Europe)
- Increase in the network supply/demand and failure of lines can cause transition to incoherency.
- Loss of frequency synchronization leads to blackouts.

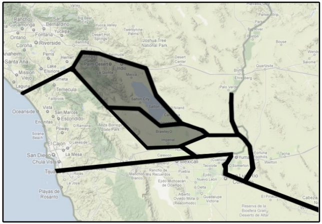


Figure: Southern California Blackout 2011– the gray area is the islanded part of the grid

Phenomenon #2: Multi-stable power flows

Theoretical observation:

- Multiple stable operating points exist in power networks
  [A. Korsak, On the question of uniqueness of stable load-flow solutions, 1972]

ON THE QUESTION OF UNIQUENESS OF STABLE LOAD-FLOW SOLUTIONS

Andrew J. Korsak

Stanford Research Institute
Menlo Park, California

Abstract – Practical experience with load-flow solutions has indicated that stable solutions are probably unique, given sufficient data about a network to determine a solution from the standpoint of dimensionality, i.e., having as many equations as variables for the remaining unknowns. A counterexample is given here for which two stable solutions exist, and an approach towards the analysis of uniqueness is presented that adds some insight into the nature of load-flow solutions in general.

1. INTRODUCTION

This paper exhibits a counterexample to the expectation expressed in Ref. 1 that load-flow problems should possess a unique (stable) solution.* In addition, a general approach is presented that adds insight into the nature of load-flow solutions. The possibility of more than one stable load-flow solution has some implication on the method such as in Refs. 2 and 3 for determining transient stability of

P_i = real power injection to network at node i
Q_i = reactive power injection to network at node i
A_i = set of nodes j having a branch connecting them to node i in the network
n = number of nodes in network.

Other equivalent formulations of the load-flow problems are possible, employing complex variables, as well as adding admittances to ground at the nodes, etc., but the above is preferable for what follows.

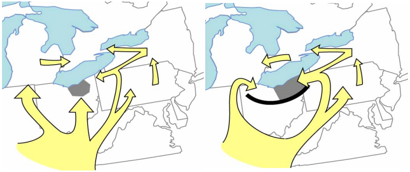
To specify a solution, some two of the four quantities V, δ, P, and Q must be given at each node, or else some two additional algebraic relations among two or more of these variables must be stated, such as those resulting from a constant (or current dependent) impedance to ground at a node. If P_i and V_i are given, node i is referred to as "PV node," etc.

3. STABILITY OF LOAD-FLOW SOLUTIONS

Practical observations:

1. Undesirable power flows around loops (Fig. (a): Lake Erie loop in the 2009 US blackout)
2. Dramatic change in power flow patterns before and after line trip (Fig. (b): Sammis–Star Trip in the 2003 US blackout)

(a)



(b)

Model: Coupled Oscillators Network

Pendulum clocks: "an odd kind of sympathy"
[C. Huygens, Horologium Oscillatorium, 1673]

Models for coupled oscillators:
[A. T. Winfree, 1967 and Y. Kuramoto 1975]

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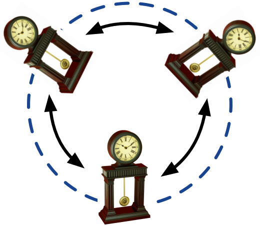

Kuramoto Model

1. **n-oscillators** with phases θ_i,
2. with natural frequencies $\omega_i \in \mathbb{R}$,
3. **coupling** with strength $a_{ij} = a_{ji}$.

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j).$$

Model: Active Power Dynamics

1. **generators** ■ and **inverters and loads** ●

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      active power p_i

Model: Active Power Dynamics

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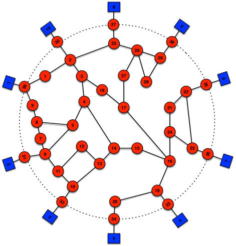
2. **physics:**
   1. Kirchhoff and Ohm laws
   2. **quasi-sync:** voltage and phase V_i, θ_i
      active power p_i
3. simplifying assumptions:
   1. lossless and inductive lines with admittances Y_{ij}
   2. decoupling of phase and voltage dynamics



New England IEEE 39-bus

Model: Active Power Dynamics

Structure-Preserving Model [A. Bergen & D. Hill, 1981]:

Active Power Dynamics

Generators: $M_i \ddot{\theta}_i + D_i \dot{\theta}_i = p_i - \sum_j a_{ij} \sin(\theta_i - \theta_j)$

Inverters: $\Lambda_i \dot{\theta}_i = p_i - \sum_j a_{ij} \sin(\theta_i - \theta_j)$

Loads: $\tau_i \dot{\theta}_i = p_i - \sum_j a_{ij} \sin(\theta_i - \theta_j)$

where

Active power capacity of line (i, j): $a_{ij} = |Y_{ij}| V_i V_j$

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Synchronization= sync'd frequencies & bounded active power flows

$$\dot{\theta}_i = \omega_{\mathrm{sync}},\ \forall \text{ bus } i \quad \& \quad |\theta_i - \theta_j| \leq \gamma < \tfrac{\pi}{2},\ \forall \text{ line } (i,j)$$

Synchronization problem

Synchronization = Equilibrium point = Stable operating point

$$p_i = \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j), \ \ \forall \text{ bus } i,$$

$$|\theta_i - \theta_j| \leq \gamma < \frac{\pi}{2}, \ \ \forall \text{ line } (i,j)$$

Key questions

Given the network and the power profile:

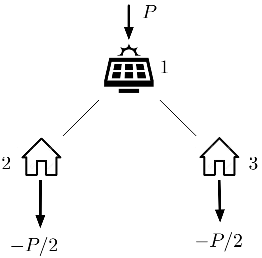
Q1: does there exist a **stable operating point**?

Q2: is the stable operating point **unique**?

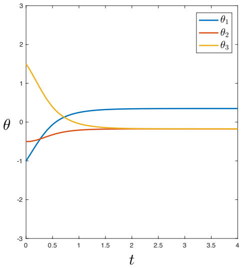
Q3: how to measure the **robustness** of the synchronization?

Q1: Existence of an operating point:

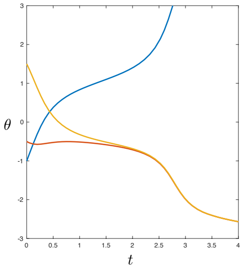
$$\dot{\theta}_i = p_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$



$a_{ij} = 1$



$P = 1$



$P = 2.5$

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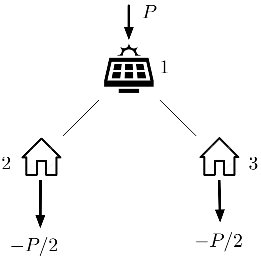

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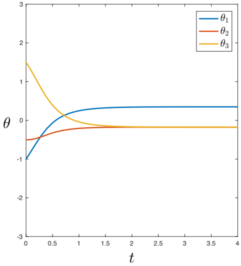
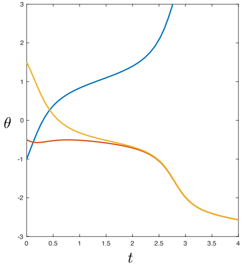


$P = 1$



$P = 2.5$

- sync threshold : "power transmission" vs. "coupling"
- quantify: "power transmission" $<$ "coupling"
- as a function of network parameters

Primer on algebraic graph theory

Weighted undirected graph with n nodes and m edges:

Incidence matrix: $n \times m$ matrix B s.t. $(B^\top p)_{(ij)} = p_i - p_j$

Edge weight matrix: $m \times m$ diagonal matrix $\mathcal{A}$

Laplacian matrix: $L = B\mathcal{A}B^\top$

Operating point:

$$p = B\mathcal{A}\sin(B^\top \theta)$$

Algebraic connectivity:

$$\lambda_2(L) = \text{second smallest eig of } L$$

$$= \text{notion of connectivity and coupling}$$

Known results

Given a network and p, does there exist angles?

$$p = B\mathcal{A}\sin(B^\top \theta),$$
$$|\theta_i - \theta_j| \le \gamma < \frac{\pi}{2}, \ \ \forall \text{ line } (i,j).$$

synchronization arises if

power transmission $<$ **coupling strength**

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Equilibrium angles (neighbors within $\pi/2$ arc) exist if

$\|B^\top p\|_2 \quad < \sin(\gamma)\lambda_2(L) \qquad$ for all graphs (Old 2-norm T)

(Old ∞-norm T)

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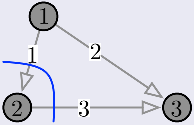
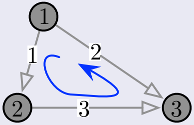
Equilibrium angles (neighbors within $\pi/2$ arc) exist if

$\lVert B^\top p\rVert_2$	$< \sin(\gamma)\lambda_2(L)$	for all graphs	(Old 2-norm T)
$\lVert B^\top L^\dagger p\rVert_\infty$	$< \sin(\gamma)$	for trees, complete	(Old ∞-norm T)

Novel: algebraic potential theory





$$\underbrace{\mathbb{R}^m}_{\text{edge space}} = \underbrace{\mathrm{Im}(B^\top)}_{\text{cutset space}} \oplus \underbrace{\mathrm{Ker}(B\mathcal{A})}_{\text{weighted cycle space}}$$

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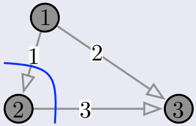

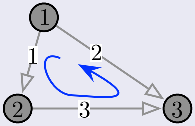

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parallel to $\mathrm{Ker}(B\mathcal{A})$

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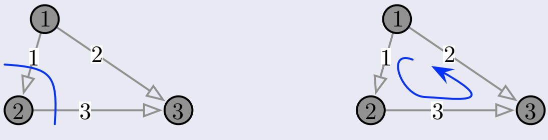

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1. if G acyclic, then $\mathcal{P} = I_m$
2. if G unweighted, then $\mathcal{P}$ is an orthogonal projection
3. if $R_{\mathrm{eff}} \in \mathbb{R}^{n\times n}$ are effective resistances, then $\mathcal{P} = -\frac{1}{2} B^\top R_{\mathrm{eff}} B \mathcal{A}$

Rewriting the equilibrium equation

Find sufficient conditions on $B, \mathcal{A}, p$ s.t. there exists a solution θ to:

$$p = B\mathcal{A}\sin(B^\top\theta),$$
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Key idea: Node vs. Edge

$$p = B\mathcal{A}\sin(B^\top\theta) \qquad \textbf{Node balance eq. } \mathbb{R}^n$$
$$\Updownarrow$$
$$B^\top L^\dagger p = \mathcal{P}\sin(B^\top\theta) \qquad \textbf{Edge balance eq. } \mathbb{R}^m$$

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- **Edge variables**: $x = B^\top \theta$ and $z = B^\top L^\dagger p$

Find sufficient conditions on $z \in \mathrm{Im}(B^\top)$ s.t. there exists solution x to:

$$z = \mathcal{P}\sin(x) = \mathcal{P}[\mathrm{sinc}(x)]x$$

Brouwer's Fixed-Point: A unifying theorem

2. look for $x \in \mathcal{B}_q(\gamma) = \{x \mid \|x\|_q \leq \gamma\}$ solving

$$\mathcal{P}[\mathrm{sinc}(x)]x = z \quad \Longleftrightarrow \quad x = (\mathcal{P}[\mathrm{sinc}(x)])^{-1}z =: h(x)$$

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$$\alpha_q(\gamma) := \min_{\|x\|_q \leq \gamma} \min_{\|y\|_q = 1} \|\mathcal{P}[\mathrm{sinc}(x)]y\|_q$$

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$$\|z\|_q \leq \gamma\alpha_q(\gamma) \quad \Longrightarrow \quad h \text{ satisfies Brouwer on } \mathcal{B}_q(\gamma)$$

Brouwer's Fixed-Point: A unifying theorem

Equilibrium angles (neighbors within γ arc) exist if, in some q-norm,

$$\|B^\top L^\dagger p\|_q \leq \gamma \alpha_q(\gamma) \qquad \text{for all graphs} \qquad \text{(New } q\text{-norm T)}$$

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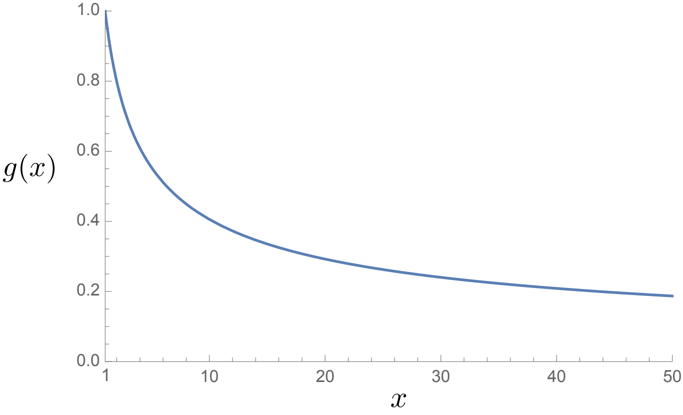
For $q = \infty$, the new test for all graphs

$$\|B^\top L^\dagger p\|_\infty \leq g(\|\mathcal{P}\|_\infty) \qquad \text{(New } \infty\text{-norm T)}$$

Function g is strictly decreasing



$g : [1, \infty) \to [0, 1]$

$$g(x) = \frac{y(x) + \sin(y(x))}{2} - x\frac{y(x) - \sin(y(x))}{2} \Big|_{y(x)=\arccos(\frac{x-1}{x+1})}$$

Comparison of synchronization tests

K_C = critical coupling of Kuramoto model, computed via MATLAB *fsolve* K_T = smallest value of scaling factor for which test T fails

Test Case	Critical ratio K_T/K_C			
	Old 2-norm	New ∞-norm $g(\lVert\mathcal{P}\rVert_\infty)$	Old ∞-norm Approx.test	New ∞-norm $\alpha_\infty(\pi/2)$
IEEE 9	16.54 %	73.74 %	92.13 %	85.06 %†
IEEE RTS 24	3.86 %	53.44 %	89.48 %	89.48 %†
New England 39-bus	2.97 %	67.57 %	100 %	100 %†
IEEE 118	0.29 %	43.70 %	85.95 %	—*
IEEE 300	0.20 %	40.33 %	99.80 %	—*
Polish 2383	0.11 %	29.08 %	82.85 %	—*

† *fmincon* has been run for 100 randomized initial phase angles.
* *fmincon* does not converge.

Old 2-norm:
$\lVert B^\top p\rVert_2 \leq \sin(\gamma)\lambda_2(L)$

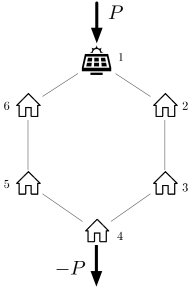
New ∞-norm:
$\lVert B^\top L^\dagger p\rVert_\infty \leq g(\lVert\mathcal{P}\rVert_\infty)$

Old ∞-norm Approx.:
$\lVert B^\top L^\dagger p\rVert_\infty \leq \sin(\gamma)$

New ∞-norm:
$\lVert B^\top L^\dagger p\rVert_\infty \leq \frac{\pi}{2}\alpha_\infty(\frac{\pi}{2})$

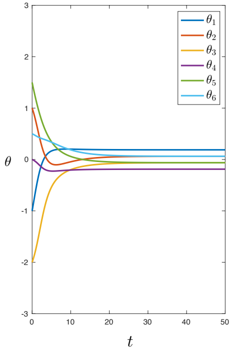
Q2: Is the operating point unique?

$$\dot{\theta}_i = p_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

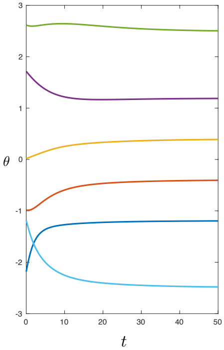


$a_{ij} = 1$

$P = 1/4$



$\theta_0 = [-1, 1, -2, 0, 1.5, 0.5]^\top$



$\theta_0 = [0, 1.2, 2.2, 3.9, 4.8, 1]^\top$

Q2: Is the operating point unique?

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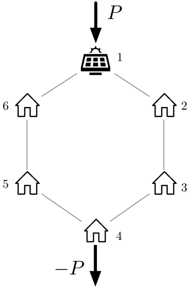

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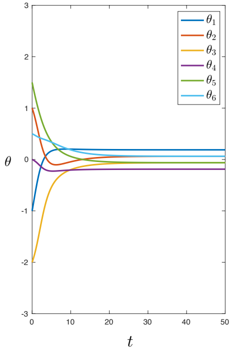
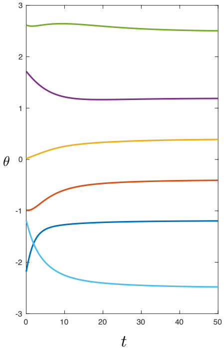


$\theta_0 = [-1, 1, -2, 0, 1.5, 0.5]^\top$



$\theta_0 = [0, 1.2, 2.2, 3.9, 4.8, 1]^\top$

- multi-stable sync : "cycle structure" and "state space"
- quantify: "cycle structure" vs "multi-stable sync"

Algebraic graph theory on n-torus

Key question

How to localize stable operating points?

Winding number

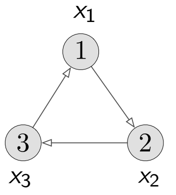
Algebraic graph theory on n-torus

Key question

How to localize stable operating points?

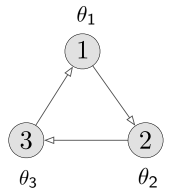
Winding number

Nodal variables in $\mathbb{R}^3$



$$\sum_{i=1}^{3} \overbrace{(x_i - x_{i+1})}^{\text{distance in } \mathbb{R}} = 0.$$

Nodal variables in $\mathbb{T}^3$



$$\sum_{i=1}^{3} \overbrace{(\theta_i - \theta_{i+1})}^{\text{distance in } \mathbb{S}} = 2\pi w_\sigma(\theta),$$

$w_\sigma(\theta) \in \mathbb{Z}$, winding number

Winding vector

Given a graph G with a cycle basis $\Sigma = \{\sigma_1, \dots, \sigma_{m-n+1}\}$ and $\theta \in \mathbb{T}^n$.

Winding vector: $$\mathbf{w}_\Sigma(\theta) = [w_{\sigma_1}(\theta), \dots, w_{\sigma_{m-n+1}}(\theta)]^\top$$

Theorem: Kirchhoff angle law on $\mathbb{T}^n$

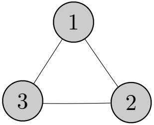
$$w_\sigma(\theta) = 0, \pm 1, \dots, \pm\lfloor n_\sigma/2 \rfloor$$
$$\implies \quad \mathbf{w}_\Sigma(\theta) \quad \text{is piecewise constant,}$$
$$\mathbf{w}_\Sigma(\theta) \quad \text{takes value in a finite set}$$

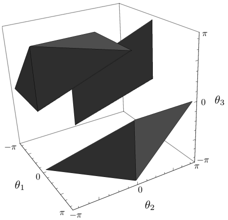
Winding partition of the n-torus
Winding cells

Winding cells: equivalence classes

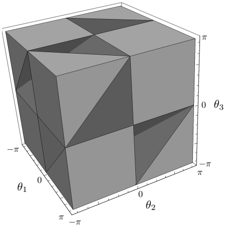
Given a graph G with a cycle basis Σ. For every $\mathbf{u} \in \mathbb{Z}^{m-n+1}$

$$(\text{Winding cell } \mathbf{u}) \quad = \quad \text{all } \theta \in \mathbb{T}^n \text{ s.t. } \mathbf{w}_\Sigma(\theta) = \mathbf{u}.$$





$\mathbf{u} = -1$



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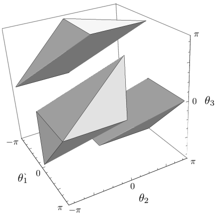
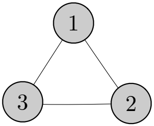


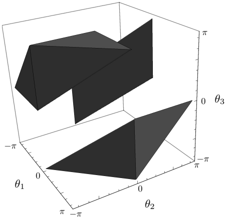
$\mathbf{u} = +1$

Theorem: Winding partition of n-torus

$$\mathbb{T}^n = \bigcup_{\mathbf{u} \in \mathbb{Z}^{m-n+1}} (\text{Winding cell } \mathbf{u})$$

Properties of the winding partition





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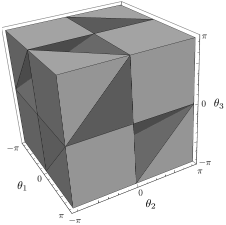

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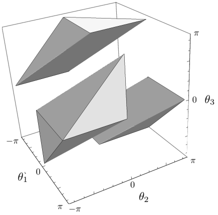
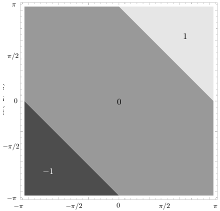


$\mathbf{u} = +1$

- each winding cell is connected
- each winding cell is invariant under rotation
- **bijection**: winding cell $\longleftrightarrow$ convex polytope

$$\|B^\top \theta - 2\pi C_\Sigma^\dagger \mathbf{u}\|_\infty < \pi.$$

C_Σ is the cycle-edge incidence matrix



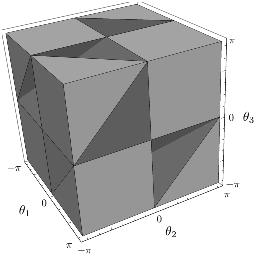
At-most uniqueness in winding cells

Multi-stability

Given topology, admittances, and injections

$$p_i = \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j), \ \ \forall \text{ bus } i$$

$$|\theta_i - \theta_j| \leq \gamma < \frac{\pi}{2}, \ \ \forall \text{ line } (i,j)$$



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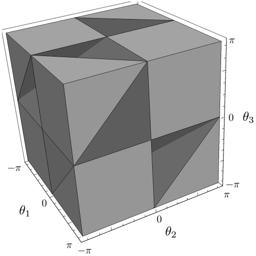

Theorem: At-most-uniqueness and extensions

At-most uniqueness in winding cells

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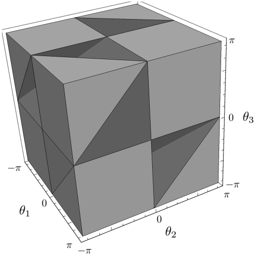

Theorem: At-most-uniqueness and extensions

1. each winding cell has at-most-unique equilibrium with $\Delta\theta \leq \gamma$

At-most uniqueness in winding cells

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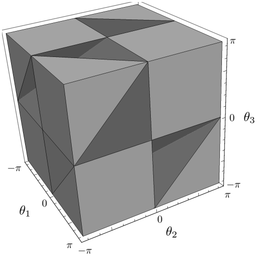
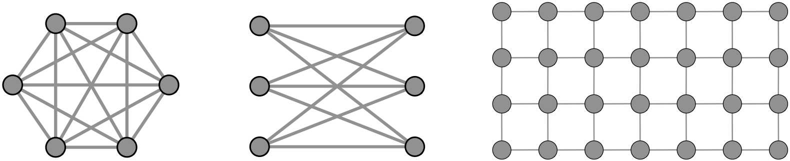


Theorem: At-most-uniqueness and extensions

1. each winding cell has at-most-unique equilibrium with $\Delta\theta \leq \gamma$
2. equilibrium loop flow increases monotonically wrt winding number



- **Complete graph:** cycle basis of lenght 3 $\Longrightarrow$ at most one equilibrium
- **Complete bipartite:** cycle basis of lenght 4 $\Longrightarrow$ at most one equilibrium
- **Rectangular grid:** cycle basis of lenght 4 $\Longrightarrow$ at most one equilibrium

Iterations for normalized power flows

$$\eta^{(k+1)} = \eta^{(k)} - \cos(\gamma)(I_n - \mathcal{P})\big(\mathrm{asin}(\eta^{(k)}) - 2\pi C_{\Sigma}^{\dagger}\mathbf{u}\big).$$

- Start from $\eta^{(0)} = \mathcal{A}^{-1}B^{\top}L^{\dagger}p$
- The sequence is **contractive** and always converges (to a vector η^{*})
- **If** $\|\eta^{*}\|_{\infty} > \sin(\gamma)$: no stable equilibrium point in winding cell $\mathbf{u}$.
- **If** $\|\eta^{*}\|_{\infty} \leq \sin(\gamma)$: one stable equilibrium point in winding cell $\mathbf{u}$;

$$\theta^{*} = L^{\dagger}B\mathcal{A}(\mathrm{asin}(\eta^{*}) - 2\pi C_{\Sigma}^{\dagger}\mathbf{u})$$

Summary

- geometry of cutset projection operator
- family of sufficient sync conditions
- partiton of n-torus based on winding vector
- localize the equilibrium points using winding cells

Future research

- close the gap between sufficient and necessary conditions
  - use suitable optimization algorithms to compute min amplification factor.
- region of attraction of stable equilibrium points
  - design suitable Lyapunov functions which can ensure convergence with transient guarantees
- generalizations to other oscillator models.
  - extension to multi-stability of state-space coupled oscillators such as FitzHugh–Nagumo