

THERMOELASTIC DAMPING IN THE LONGITUDINAL VIBRATION: ANALYSIS AND SIMULATION

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ABSTRACT

The thermoelastic damping (TED), one of the main energy lost sources in MEMS and NMEMS, is analyzed for the longitudinal vibration mode. Two approximations based on the insulated thermal boundary condition (B.C.) and the modified fixed temperature B.C. are developed and compared. Until now, the highest reported experimental Q factors for the longitudinal vibrating MEMS device are still more than one order less than the predicted TED Q factors. Therefore, a numerical simulation approach is proposed and used for validation. The aim of this paper is to provide a prediction of TED for the fast developing bulk mode micro and sub-micro oscillators operating in a longitudinal vibrating mode.

1. INTRODUCTION

In Micro and Nano Electro-Mechanical Systems (MEMS & NEMS), flexural-mode resonators, such as cantilevers and membrane resonators, are widely designed and implemented in low and mid frequency range applications, and the corresponding high Q factors have been achieved [1-4]. For high frequency applications, bulk-mode resonators are becoming more common. One main advantage is the relatively large size when compared to flexural-mode resonators at the same frequency. It is consistently observed that the Q factor usually increases with the increasing of the dimensions [4].

Thermoelastic damping theory, especially for the flexural-mode vibration, has been systematically developed from 1930's. Zener [5], investigated the thermoelastic internal friction on the basis of the one-dimensional theory of elasticity. And a well-known approximation is given for the vibrating reeds. A.S. Nowick [6] presented a general description of TED for a second-order mechanical system. V.Kinra, K Milligan [7] and Ron Lifshitz [8] reported different approaches seeking the exact solution of thermo-elastic damping for the flexural mode vibration. For microscale devices, Amy Duwel [9] reported that the experimental results on MEMS gyros showed a good agreement with Zener's approximation.

Comparing to the advanced development of the thermo-elastic damping analysis for the flexural vibration mode, the thermoelastic damping analysis for the longitudinal vibration mode is paid relatively little attention. Landau and Lifshitz described the absorption of sound in solids for both transverse and longitudinal waves [10], with the assumption that sound oscillations are adiabatic as the first approximation. A.S. Nowick's [6] general description of thermo-elastic damping for a generalized second-order mechanical system is elaborated in detail for the flexural mode, but no detailed discussion for the longitudinal mode.

The aim of this paper is to present a detailed description of TED and to give a TED prediction in the longitudinal vibrating mode. Two approximations based on the insulated thermal boundary condition (B.C.) and the modified fixed temperature thermal B.C. are developed and compared, following the general way Zener defined the Q factor for the general second order mechanical system. Process of the numerical simulation is presented in detail and the result is used to validate the theoretical analysis. First we start from a review of Zener's general process of TED.

2. REVIEW: ZENER'S GENERAL PROCESS OF TED

Zener [5] gave a general process to define the damping for a general second order continuous system with the governing partial differential equation:

$$\frac{\partial^2 \vec{u}(\vec{r}, t)}{\partial t^2} + L\vec{u}(\vec{r}, t) = \vec{f} \quad (1)$$

Where $\vec{u}(\vec{r}, t)$ denotes the displacement vector field. The first term is the inertial force on unit mass. The second term ($L\vec{u}$) represents the force due to the elasticity of the solid. L is an operator acting on position \vec{r} . \vec{f} denotes the body force per unit mass, including the force that arises from the temperature

fluctuation. To examine the harmonic solution of (1), \vec{u} and \vec{f} can be written as:

$$\begin{aligned}\vec{u}(\vec{r}, t) &= U(\vec{r})e^{i\omega t} \\ \vec{f}(\vec{r}, t) &= F(\vec{r})e^{i\omega t}\end{aligned}\quad (2)$$

Where ω is the angular frequency. Substitute (2) into (1), we get:

$$(L - \omega^2)U = F \quad (3)$$

In general, U and F are not in phase with each other. It is this phase difference that gives rise to the damping. We can always choose the origin of the time so that U is real. Then in general, F is a complex force with the form:

$$F = F_1 + iF_2 \quad (4)$$

And further let U_0 be the solution of (3) when F_2 is neglected, which means no damping. Usually the damping force F_2 has only a small effect on the solution of (3) comparing to the elastic force LU , it is a proper approximation that U_0 is used to calculate the energy stored in the solid and the dissipated energy per cycle. The damping is then defined as:

$$Q^{-1} = \left| \int U_0 \cdot F_2 dv \right| / \left| \int U_0 \cdot LU_0 dv \right| \quad (5)$$

The numerator stands for energy loss per cycle and the denominator stands for the stored energy in the solid. In the later sections, we will follow Zener's process to define the thermoelastic damping and to develop an approximation for the longitudinal vibration mode.

3. LINEAR THERMO-ELASTICALLY COUPLED MODEL FOR THE LONGITUDINAL VIBRATION MODE.

For the linear thermal-elasticly coupled model, the governing equations are [6, 11]:

$$\begin{aligned}\rho \frac{\partial^2 \vec{u}}{\partial t^2} &= \nabla \cdot C(\nabla \vec{u} - \vec{\alpha}T) \\ C_v \frac{\partial T}{\partial t} &= \nabla \cdot \kappa \nabla T - T_0 \vec{\alpha}^T C \nabla \vec{u}\end{aligned}\quad (6)$$

Where $\vec{u}(\vec{r}, t)$ is the displacement vector, T is the temperature deviation from the reference temperature T_0 . $\vec{\alpha}$ is the linear thermal expansion coefficient vector, ρ is the density, C is the stiffness tensor. $C_v = \rho C_{sp}$ with

C_{sp} as the specific heat with constant pressure. κ is the thermal conductivity of the material.

Now we consider the one dimensional longitudinal vibration mode (Figure 1). The bar-shaped oscillator vibrates along x-axis symmetrically with $x=0$. The oscillator has a total length a .

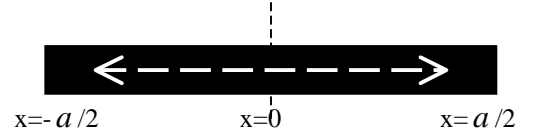


Figure 1 : longitudinal vibrating bar with length a: symmetric along $x=0$

Only x-direction movement with displacement $u(x, t)$ is considered in (6). For the one dimensional model, the coupled governing equations have the form:

$$\begin{aligned}\rho \ddot{u} &= Cu_{xx} - C\alpha T_x \\ \dot{T} &= DT_{xx} - \beta \dot{u}_x\end{aligned}\quad (7)$$

with $D = \kappa/C_v$, $\beta = T_0 \alpha C/C_v$

The mechanical boundary conditions are:

$$u_x - \alpha T = 0 \quad \text{at} \quad x = \pm a/2 \quad (8)$$

The insulated thermal boundary conditions are

$$T_x = 0 \quad \text{at} \quad x = \pm a/2 \quad (9)$$

The symmetric movement condition can be expressed as:

$$u = 0, T_x = 0 \quad \text{at} \quad x = 0 \quad (10)$$

With the equations (7) and the boundary conditions (8-10), we develop an approximation for the thermoelastic damping based on Zener's approach mentioned before.

4. APPROXIMATIONS WITH TWO DIFFERENT THERMAL BOUNDARY CONDITIONS

The difficulty of using expression (5) is to find the damping force F_2 , which is caused by the thermoelastic coupling in TED. The thermal boundary conditions will play a very important role in finding the temperature distribution form, which determines the form of F_2 . First we start from the insulated thermal boundary conditions. An approximation is presented and discussed. A second approximation is then derived basing on the modified thermal boundary conditions.

4.1 Approximation for the insulated thermal B.C.

It is not difficult to find the function of U_0 , the undamped displacement function, which is a simple trigonometric sine function. To satisfy the insulated thermal boundary conditions (9), the temperature function T may have the form of a trigonometric series. With this discussion and the assumption that only the harmonic solution is considered, we have the following expressions of u and T :

$$u = U_0 e^{i\bar{\omega}t} = A \sin(\pi x / a) e^{i\bar{\omega}t} \quad (11)$$

$$T = \sum_{k=0}^{\infty} b_k \cos(2k\pi x / a) e^{i\bar{\omega}t} \quad (12)$$

A is the corresponding vibration amplitude. With expression (11) and (12), we can calculate the damping force F_2 and the Q factor due to the thermoelastic damping using expression (5). By substituting (12) into the second equation of (7), we get the expression of the temperature T in terms of the function of U_0 . Here is the result:

$$b_k = c_k \frac{i\bar{\omega}}{i\bar{\omega} + \mu_k} = c_k \frac{i\bar{\omega}(\mu_k - i\bar{\omega})}{\bar{\omega}^2 + \mu_k^2} \quad (13)$$

with

$$\mu_k = D \left(\frac{2k\pi}{a} \right)^2, \quad c_k = (-1)^{k+1} \frac{4\beta A}{a(1-4k^2)} \quad (14)$$

Substitute (14) into the first equation of (7) and do the integration, we get:

$$\begin{aligned} \rho \bar{\omega}^2 &= \left(\frac{\pi}{a} \right)^2 C + C\alpha\beta \left(\frac{\pi}{a} \right)^2 \sum_{k=0}^{\infty} \left(\frac{8k}{\pi(1-4k^2)} \right)^2 \frac{i\bar{\omega}(\mu_k - i\bar{\omega})}{\bar{\omega}^2 + \mu_k^2} \\ &\approx \left(\frac{\pi}{a} \right)^2 C + C\alpha\beta \left(\frac{\pi}{a} \right)^2 \sum_{k=0}^{\infty} \left(\frac{8k}{\pi(1-4k^2)} \right)^2 \frac{i\bar{\omega}\mu_k}{\bar{\omega}^2 + \mu_k^2} \end{aligned} \quad (15)$$

Here the first term is $\left| \int U_0 \cdot LU_0 dv \right|$, and the absolute value of the second term is $\left| \int U_0 \cdot F_2 dv \right|$. Then the thermoelastic damping could be written out using (5). In general the angular frequency $\bar{\omega}$ is a complex value with $\bar{\omega} = \bar{\omega}_r + i\bar{\omega}_i$. From (5) and (15), we get the approximation of TED as:

$$Q_{TED}^{-1} = 2 \frac{\bar{\omega}_i}{\bar{\omega}_r} = \alpha\beta \sum_{k=0}^{\infty} \left(\frac{8k}{\pi(1-4k^2)} \right)^2 \frac{\bar{\omega}_r \mu_k}{\bar{\omega}_r^2 + \mu_k^2} \quad (16)$$

The above approximation is based on expression (12), which automatically satisfies the insulated thermal boundary

conditions. But note that expression (11) and (12) do not satisfy the mechanical boundary conditions (8). The error of this violation of the mechanical boundary conditions will be shown later. Next we will seek another approximation by modifying the boundary conditions.

4.2 Approximation for the fixed temperature thermal B.C.

Unlike the flexural vibrating mode in which the maximum amplitude of the strain and the maximum temperature amplitude appear on the upper and lower boundaries. In the longitudinal vibrating mode, the maximum amplitude of the strain appears at $x=0$. On the two ends, the strain amplitude is negligible, and so is the resultant temperature. In this case, it is proper to make the assumption that the temperature on the boundary is equal to the temperature of the surroundings, which is zero.

$$T = 0 \quad \text{at} \quad x = \pm a/2 \quad (17)$$

Similar to (11) and (12), only the harmonic solution is considered. We can write the expression of u and T as:

$$u = U_0 e^{i\bar{\omega}t} = A \sin(\pi x / a) e^{i\bar{\omega}t} \quad (18)$$

$$T = \sum_{k=0}^{\infty} b_k \cos((2k+1)\pi x / a) e^{i\bar{\omega}t} \quad (19)$$

With assumption (17), the observation is that (18) and (19) satisfy both the mechanical and thermal boundary conditions.

Actually with expressions (18) and (19) we will see that the process to find the approximation is much easier, because only the first term of (19) is nonzero, which means the temperature function is a simple trigonometric cosine function. To get this we substitute (18) and (19) into the second equation of (7) and carry out the integration.

$$b_0 = -\beta A \frac{\pi}{a} \frac{i\bar{\omega}}{i\bar{\omega} + \mu_0}, \quad b_k = 0 \quad k \geq 1 \quad (20)$$

$$\text{with} \quad \mu_0 = D(\pi/a)^2$$

Substitute (18)-(20) into the first equation of (7) and do the integration, we get:

$$\begin{aligned} \rho \bar{\omega}^2 &= \left(\frac{\pi}{a} \right)^2 C \left(1 + \alpha\beta \frac{i\bar{\omega}}{i\bar{\omega} + \mu_0} \right) \\ &\approx \left(\frac{\pi}{a} \right)^2 C \left(1 + \alpha\beta \frac{i\bar{\omega}\mu_0}{\mu_0^2 + \bar{\omega}^2} \right) \end{aligned} \quad (21)$$

Similar to (16), the approximation of the TED is:

$$Q_{TED}^{-1} = 2\bar{\omega}_i / \bar{\omega}_r = \alpha\beta \frac{\bar{\omega}_r \mu_0}{\bar{\omega}_r^2 + \mu_0^2} \quad (22)$$

4.3 Discussion

Both the approximations (16) and (22) have similar form to Zener's approximation for the flexural mode:

$$Q_{TED}^{-1} = \alpha\beta \sum_{k=0}^{\infty} \left(\frac{96}{\pi^4} \frac{1}{(1+2k)^4} \right) \frac{\bar{\omega}_r \mu_k}{\mu_k^2 + \bar{\omega}_r^2} \quad (23)$$

with a as the flexural width, and $\mu_k = D \left(\pi \frac{2k+1}{a} \right)^2$.

Usually the first term of (23) is used for the approximation, which is almost the same as expression (22).

Now we compare the two approximations (16) and (22). Single crystal silicon is used with the following material properties [9]:

Density ρ	2.33e3 kg.m ⁻³	Specific heat C_{sp}	713 J.kg.K ⁻¹
Thermal Conductivity κ	156 W/m.K	Thermal Expansion α	2.59e-6 K ⁻¹
Stiffness C	1.69e11 N.m ⁻²	Reference temperature T_0	290 K

Table 1: Mechanical and thermal properties of Silicon

We pick some typical length scales in MEMS and NEMS and calculate the Q_{TED} factor for both (16) and (22).

Length a (um)	Q_{TED} from Approximation (16)	Q_{TED} from Approximation (22)
0.5	1.8711e+004	7.3290e+004
5	6.2527e+004	7.2944e+005
50	2.0746e+005	7.2940e+006
500	6.6825e+005	7.2940e+007
1000	9.4753e+005	1.4588e+008

Table 2: Comparison of (16) and (22) for different length a

From Table 2, we see that for the typical length scale in MEMS and NEMS, the thermoelastic damping approximation (16) differs from the result in (22) greatly, which means the mentioned violation of the mechanical boundary condition causes a very big error in the approximation (16).

Next we need to investigate the error of the approximation of (22) caused by modifying the thermal boundary conditions from (9) to (17). This is achieved by numerical simulation.

5. NUMERICAL SIMULATION

The numerical simulation is important for two reasons. First it can be used to check the validity of the analytical result. Second, for complex geometry and multiple materials, there is no simple approximation available to estimate the TED, in which case, numerical simulation is necessary. B.Antkowiak and A.E. Duwel [11] pioneered the thermoelastic simulation in MEMS. Here we elaborate the simulation process in a succinct way based on matrix manipulation.

Let $\bar{v} = \partial \bar{u} / \partial t$. We can rewrite the equations (6) as:

$$d_1 \dot{U} = \nabla \cdot (C_1 \nabla U + \alpha_1 U) - a_1 U - \beta_1 \cdot \nabla U \quad (24)$$

with

$$U = \begin{pmatrix} \bar{u} \\ \bar{v} \\ T \end{pmatrix} \quad d_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & C_v \end{pmatrix} \quad C_1 = \begin{pmatrix} 0 & 0 & 0 \\ C & 0 & 0 \\ 0 & 0 & \kappa \end{pmatrix}$$

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -C\bar{\alpha} \\ 0 & 0 & 0 \end{pmatrix} \quad a_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \beta_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & T_0 \bar{\alpha}^T C & 0 \end{pmatrix}$$

The platform of the numerical simulation is FEMLAB software, in which the coefficient form PDE is chosen. The calculation of the eigen values is carried out to get the angular frequency $\bar{\omega}$ ($\bar{\omega} = \bar{\omega}_r + i\bar{\omega}_i$), where the real part stands for the mechanical resonant frequency. The thermoelastic damping is then defined as:

$$Q_{TED}^{-1} = |2\bar{\omega}_i / \bar{\omega}_r| \quad (25)$$

We simulate the TED not only for the insulated thermal boundary conditions, but also for the modified fixed temperature thermal boundary conditions as a comparison. Using the same material parameters in table (1) and the length scale parameters in table (2), the simulation is carried out using 1-dimensional model. The results are listed in table (3).

Length a (μm)	Q_{TED} with insulated thermal B.C.	Q_{TED} with fixed temperature thermal B.C.
0.5	1.0004e+005	7.3312e+004
5	7.8963e+005	7.2966e+005
50	7.4744e+006	7.2971e+006
500	7.3515e+007	7.2962e+007
1000	1.4670e+008	1.4592e+008

Table 3: TED FEMLAB simulation results with the different length scales for the insulated thermal B.C. and the fixed temperature thermal B.C.

From Table 3, we can tell that for typical length scale in MEMS ($>1\mu\text{m}$), the error between the TED results from the two different thermal boundary is small ($<10\%$). And the error decreases with the increasing of the length scale. These results support the modification of the thermal boundary condition in (17). Comparing Table 2 with Table 3 also validates the TED assumption (22). However, in sub-micron region, the error could be big. For example error is about 30% at $a=0.5\mu\text{m}$.

6. CONCLUSION AND DISCUSSION

We elaborate the process to derive an approximation of the thermoelastic damping for the longitudinal vibration mode. Starting from the insulated thermal boundary conditions, we get the first approximation with big error due to the violation of the mechanical boundary conditions. By modifying the thermal boundary conditions, we derive a simple approximation of the TED. Numerical simulation process is set up for validation, which shows the validity of the modification of the thermal boundary conditions and the corresponding TED approximation for the typical length scale in MEMS.

Despite the fast development in the bulk mode resonators, such as FBAR [12] and disk resonators [13], the Q factors from the experiment for the bulk mode resonators [14] till now are still more than one order less than the predicted thermoelastic Q factors. Other damping sources, for example the support damping, may play important roles in the overall damping. This is currently under further investigation.

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