Synchronization of connected oscillator networks under global and local cues is ubiquitous in both science and engineering. Over the last few decades, enormous attention has been paid to study synchronization conditions of connected oscillators in chemistry, physics, mechanics, and particularly in biology. However, the influences of global and local cues on the rate of synchronization have not been fully studied. It is widespread that synchronization is achieved in the simultaneous presence of both global and local cues, such as intercellular coupling signals and external entrainment signals in terms of biological oscillators, and inter-neighbor coupling signals between follower nodes and central guiding signals in terms of groups of mobile autonomous agents. We prove in this paper that the strength of the global cue is the only determinant of the rate of synchronization. More specifically, we prove that a stronger global cue means a faster rate of synchronization whereas a stronger local cue does not necessarily make the synchronization rate faster. Our results not only apply to the noise-free case, but also apply to the case that the oscillator natural frequencies are subject to white noise. The analysis does not require the interplay to be symmetric or balanced. Simulation results are given to illustrate the proposed results.

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1. Introduction

Synchronization of oscillating dynamical systems is ubiquitous in science, nature, engineering and social life. Over the last several decades, it has been intensively studied in chemical processes, mechanical systems, physics and telecommunications, etc (Kuramoto, 1984; Pikovsky, Rosenblum, & Kurths, 2001; Wu, 2002). Recently, with the development of the emerging field of ‘systems biology’, synchronization phenomena in bioscience are also attracting considerable attention in the control community. Elucidation of the essential principles or fundamental mechanisms of synchronization has been greatly expedited by systems and control theory over the last few decades (c.f. Angeli and Sontag (2008), Arcak (2007), Jadbabaie, Lin, and Morse (2003), Kim and Allgöwer (2007), Li, Wang, and Chen (2004), Liu, Duan, Chen, and Huang (2009), Nedić, Olshevsky, Ozdaglar, and Tsitsiklis (2009), Stan and Sepulchre (2007), Xiao, Wang, Chen, and Gao (2009), Zhou, Lu, and Lu (2006) and the references therein).

The interplay of global cues and local cues (or alternatively the interplay of external cues and local coupling) is an important feature in the achievement of synchronization. For example, in mammalian circadian systems, the circadian pacemaker controlling daily rhythms is located in the suprachiasmatic nucleus (SCN), in which about 20,000 neuronal oscillators produce an extremely accurate 24-hour rhythm utilizing intercellular interplay among individual oscillators while at the same time receiving a global driving cue such as the sunlight via the retinal input (Herzog, 2007; To, Henson, Herzog, & Doyle, 2007). In the reproduction of certain species of corals, spawning is found to occur synchronously with each other within a few hours once a year, which is guided by global drivers such as ocean water temperature and phase of the moon, and simultaneously by local protein-based interactions (Guest, Baird, Clifton, & Heyward, 2008). There are many other examples of synchronization in biology, such as somatogenesis, cardiac function, respiration and insulin secretion, glycolytic oscillations and eukaryotic cell cycle, to name only a few (Goldbeter, 1997). In engineering applications, such as coordination of groups of mobile autonomous agents (e.g. UAV-unmanned aerial vehicles and MANET-mobile ad hoc networks), the signal from central resources (e.g. leader node and satellite) acts as the global cue and the interplay between the follower nodes acts as the local cue, both of which are essential to the synchronization of the collective motion (Jadbabaie et al., 2003; Sepulchre, Paley, & Leonard, 2007).
In the clock synchronization of wireless sensor networks, different
time references in a real-time network are synchronized through
internal interplay between different nodes and external coordina-
tion from the global time base such as a Global Positioning System
(GPS) (Kopetz & Oechsner, 1987). Therefore studying the influ-
ences of global and local cues in the achievement of synchroniza-
tion is not only important in scientific research, but also of crucial
importance in engineering applications.

Although synchronization of coupled oscillators has been studied
for years, compared with the rich results on synchronization
condition analysis (e.g., Ren (2007), Wieland (2010) and Wieland,
Kim, and Allgöwer (in press)) discussed the influences of global and
local cues on synchronization conditions, research on the rate of
synchronization is relatively sparse. This paper studies the influ-
ences of global and local cues on the rate of synchronization. We
prove analytically that the rate of synchronization is only deter-
bined by the strength of the global cue, and the strength of the
local cue is not directly relevant. In other words, a stronger global
cue means a faster rate of synchronization, whereas a stronger local
cue does not necessarily make the rate of synchronization faster.

2. Problem formulation

2.1. Model in the noise-free case

We first consider the phase model in the absence of noise.
Suppose the overall system is composed of a network of \( N \)
oscillators, which will henceforth be referred to as ‘nodes’. Note
that interpreted as a leader node, the global cue can be regarded as
a node in the network. We denote the phase dynamics of the global
cue as
\[
\dot{\phi}_1 = w_1
\]
and phase dynamics of the isolated oscillators as
\[
\dot{\phi}_i = w_i, \quad i = 2, 3, \ldots, N
\]
where \( \phi_1 \) and \( \phi_i (i = 2, 3, \ldots, N) \) denote the phases of the global
cue and the rest oscillators, and \( w_1 \) and \( w_i (i = 2, 3, \ldots, N) \) denote
their natural frequencies, respectively.

After taking into consideration the interplay between different
nodes, the overall dynamics can be rewritten as
\[
\dot{\phi}_1 = w_1 + \sum_{1 \leq j < N, j \neq i} a_{ij}(\phi_j - \phi_i), \quad 2 \leq i \leq N
\]
where \( a_{ij} \geq 0 \) denotes the coupling strength between node \( i \)
and node \( j \). We do not require the interplay to be symmetric, i.e., \( a_{ij} \) and \( a_{ji} \) are not necessarily equal. Note that the global cue
influences all the rest nodes but is not conversely influenced by
them. That is, \( a_{ij} (j = 2, 3, \ldots, N) \), which denote the influences
of the follower nodes on the leader node, are identically zero, but
\( a_{ij} (j = 2, 3, \ldots, N) \), which denote the influences of the leader
node on the follower nodes, are not zero.

In a biological context such as in SCN, sunlight is a global cue, and
the vasoactive intestinal peptide (VIP) is a local cue (Herzog,
2007). Therefore \( a_{ij} (j = 2, 3, \ldots, N) \) represent the influences
of sunlight on circadian neurons, and \( a_{ij} (i = 2, 3, \ldots, N) \) represent
the interplay between circadian neurons through the VIP (Herzog,
2007). And since the rhythm of sunlight is not influenced by
circadian neurons, \( a_{ij} (j = 2, 3, \ldots, N) \) are all zero.

2.2. Model in the presence of noise

When the natural frequencies \( w_i (i = 2, 3, \ldots, N) \) are subject
to noise, the dynamics can be denoted by
\[
\dot{\phi}_1 = w_1 + \sum_{1 \leq j < N, j \neq i} a_{ij}(\phi_j - \phi_i), \quad 2 \leq i \leq N
\]
where \( n_i(t) (i = 2, 3, \ldots, N) \) are Gaussian white noise terms, \( \sigma_i >
0 (i = 2, 3, \ldots, N) \) are constants. Since Gaussian white noise is
a formal derivative of a Wiener process (Karatzas & Shreve, 1991),
(4) can be rewritten as a stochastic differential equation
\[
\dot{\phi}_i = w_i \, dt + \sigma_i \, d\varphi_i(t) + \sum_{1 \leq j < N, j \neq i} a_{ij}(\phi_j - \phi_i) \, dt
\]
for \( 2 \leq i \leq N \) where \( \varphi_i(t) \) is the Wiener process.

Based on the above models, we will analyze the influences of
global cue, i.e., \( a_{ij} (2 \leq j \leq N) \) and local cue, i.e., \( a_{ij} (2 \leq i \leq N, j \neq 1) \) on the rate of synchronization.

3. Main results

3.1. Results in the noise-free case

In the noise-free case, we study the influences of global cue
\( a_{ij} (2 \leq j \leq N) \) and local cue \( a_{ij} (2 \leq j \leq N, 2 \leq i \leq N) \) on the rate of synchronization based on model (3).

Assumption 1. In the remainder of the paper, we assume that the
influences of the global cue on the rest nodes are the same, i.e.,
\( a_{21} = a_{31} = \cdots = a_{N1} = g \), where \( g \) is a positive constant.
In the case of circadian systems, the assumption means that every
neuronal oscillator is identically entrained to the same light–dark
cycle (Herzog, 2007). In the case of technological systems, this
assumption represents the situation in which the central server
(such as GPS) influences each node in the same way in the clock
synchronization of overall systems (such as computer networks
(Mills, 1995) and wireless sensor networks (Elson, 2003)).

Assumption 2. Moreover, we also assume that all the oscillators
have identical natural frequencies, i.e., \( w_1 = w_2 = \cdots = w_N = w \),
where \( w \) is a positive constant.

For convenience in the analysis, we first transform (3) into a ro-
tating reference frame via the transformation \( \psi_i = wt + \theta_i (j = 1, 2, \ldots, N) \):
\[
\dot{\theta}_i = 0
\]
\[
\dot{\theta}_i = \sum_{1 \leq j < N, j \neq i} a_{ij}(\theta_j - \theta_i), \quad 2 \leq i \leq N
\]
(6)
It can be verified that (3) and (6) have the same synchronization
properties, such as synchronization condition and the rate of syn-
chronization. Eq. (6) can be rearranged into vector form as follows
\[
\dot{\Theta} = A \Theta
\]
(7)
with \( \Theta \) and \( A \) given in (8) in Box 1. It can be easily verified that all
row–sums of matrix \( A \) are zero and ‘0’ is one of the eigenvalues of
matrix \( A \) with associated eigenvector \( [1 \ 1 \ 1]^T \). From consensus the-
ory (Olfati-Saber, Fax, & Murray, 2007; Ren, Beard, & Atkins, 2007),
it can be proved that if all the remaining eigenvalues of matrix
\( A \) have negative real parts, then all \( \theta_i \) will asymptotically achieve
agreement, i.e., the ultimate system response is
\( \theta_1 = \theta_2 = \cdots = \theta_N \), which leads to
\( \psi_1 = \psi_2 = \cdots = \psi_N \), meaning that all the oscilla-
tors are synchronized. In this paper, we will go one step further,
that is, we want to study the impact of global cues and local cues
on the rate of synchronization.

According to dynamical systems theory, the rate for all \( \theta_i \) to
achieve agreement is determined by the eigenvalues of \( A \). If the
oscillators can be synchronized, all the real parts of \( A \)’s eigenvalues
should be negative except for the unique eigenvalue ‘0’, so the
rate of synchronization is determined by the non-zero eigenvalue
with the largest real part. Therefore, we only need to study the
influences of global and local cues on the non-zero eigenvalue with
the largest real part, which is given in Theorem 1.
Theorem 1. For an oscillator network with a global cue and an arbitrary local cue (not necessarily symmetric or balanced), the oscillators will always synchronize, and the rate of synchronization is only determined by the strength of the global cue.

Proof. Multiply matrix $A$ on the right by $P$ (given in (8) in Box I on the top of the next page) and on the left by $P^{-1}$, which leads to (9) in Box II with $\tilde{A}_{2-N}$ an $(N-1) \times (N-1)$ dimensional matrix. Therefore the eigenvalues of $A$ are given by \{0, $\lambda_2$, $\lambda_3$, $\ldots$, $\lambda_N$\} where $\lambda_2$, $\lambda_3$, $\ldots$, $\lambda_N$ are the eigenvalues of $A_{2-N}$.

According to the Gershgorin Circle Theorem (Golub & Van Loan, 1996), all the eigenvalues of $\tilde{A}_{2-N}$ are no larger than $-g$. So the real parts of the eigenvalues of $A$ are no larger than $-g$ except for the eigenvalue '0'. Then according to consensus theory, the oscillators will always synchronize.

Notice that in fact $-g$ is an eigenvalue of $A$ associated with eigenvector $[0 \ 1 \ldots \ 1]^T$, which in combination with the above derivation means that $-g$ is the non-zero eigenvalue with the largest real part and it determines the synchronization rate of the oscillator network. Recall that $g$ is the strength of the global cue, thus the rate of synchronization is only determined by the strength of the global cue, which completes the proof.

Remark 1. Theorem 1 means that a stronger local cue does not necessarily bring a faster rate of synchronization, which is somewhat counterintuitive. In fact, our numerical simulations in Section 4 will confirm this point.

Remark 2. When there is no global cue, i.e., $g = 0$, $\tilde{A}_{2-N}$ in (9) reduces to the additive inverse of the graph Laplacian in Olfati-Saber et al. (2007). Therefore, we have from Olfati-Saber et al. (2007) that the network will still synchronize if the graph of the local cue is connected. Moreover, noting the fact that the maximal eigenvalue of $\tilde{A}_{2-N}$ multiplies if every $a_{ij}$ ($2 \leq i, j \leq N$) multiplies, so the rate of synchronization will increase if the strength of the local cue increases.

Remark 3. The results in Theorem 1 are equivalent to the Perron–Frobenius theory (which is limited to non-negative matrices), which asserts that a real square matrix with positive entries (also applicable to certain classes of non-negative matrices) has a unique largest real eigenvalue (Farina & Rinaldi, 2000).

Remark 4. When Assumption 1 is not satisfied, i.e., $a_{21}$, $a_{31}$, $\ldots$, $a_{N1}$ are not identical, the largest real part of the non-zero eigenvalue of $A$ is not only determined by $g$ any more. However, using a derivation similar to the proof of Theorem 1, we can obtain that the largest real part of the non-zero eigenvalue of $A$ is no larger than $-g_{\min}$ given by

$$g_{\min} = \min(a_{21}, a_{31}, \ldots, a_{N1}).$$

So we have that the rate of synchronization will still increase with an increase in the global cue.

3.2. Results in the presence of noise

Next we consider the synchronization problem in the presence of white noise in the natural frequencies. Oscillators with noise in the natural frequencies are widespread, particularly in biological oscillators such as circadian rhythms (Herzog, 2007) and quorum sensing (Wang, Li, Chen, & Aihara, 2008). Noise emerges from both molecular scale kinetics as well as environmental perturbations. Due to the presence of noise, we need to define synchronization in the stochastic framework.

Definition 1. In the presence of noise, system (5) is synchronized if the mathematical expectations of all oscillator phases asymptotically achieve agreement, and meanwhile the variances of all the oscillator phases are finite.

Similar to the noise-free case, we retain Assumptions 1 and 2, and transform (5) into a rotating reference frame first by using the transformation $\phi_j = wt + \theta_j$ ($j = 1, 2, \ldots, N$):
$d\theta_1 = 0,$ 
\[d\theta_i = \sigma_i dB_i(t) + \sum_{1 \leq j \leq N, j \neq i} a_{ij}(\theta_j - \theta_i) dt, \quad 2 \leq i \leq N \] (11)

Eq. (11) can be rearranged into vector form as follows
\[\dot{\Theta} = A\Theta + \Sigma dB(t) \] (12)
with
\[\Sigma = \text{diag}(0, \sigma_2, \ldots, \sigma_N), \quad B(t) = [0 \, B_2(t) \ldots B_N(t)]^T \]
and the rest of the parameters are the same as those in (7).

For system (12), we can also prove that the rate of synchronization is only determined by the strength of the global cue, which is given in Theorem 2.

**Theorem 2.** For an oscillator network with a global cue and an arbitrary local cue (not necessarily symmetric or balanced) formulated in (12), the oscillators will always synchronize in the sense of Definition 1, and the rate of synchronization is only determined by the strength of the global cue.

**Proof.** We first prove that the mathematical expectations of all $\theta_i$ will achieve agreement asymptotically, and then prove that their variances are finite.

According to Karatzas and Shreve (1991), (12) is an N-dimensional Ornstein–Uhlenbeck process, and the solution is given by
\[\Theta(t) = e^{At}\Theta(0) + \int_0^t e^{A(t-\tau)}\Sigma dB(\tau). \] (13)

The mathematical expectation $E[\Theta(t)]$ and variance $V[\Theta(t)]$ of $\Theta(t)$ are given by
\[E[\Theta(t)] = e^{At}E[\Theta(0)] \]
\[V[\Theta(t)] = e^{At}V[\Theta(0)]e^{At} + \int_0^t e^{A(t-\tau)}\Sigma^T \Sigma e^{A(t-\tau)} d\tau. \] (15)

First we show the mathematical expectation $E[\Theta(t)]$ will achieve agreement and the convergence rate is only determined by the strength of the global cue. $E[\Theta(t)]$ can be regarded as the solution to
\[\frac{dE[\Theta]}{dt} = AE[\Theta]. \] (16)

By using a derivation similar to that of Section 3.1, we can obtain that the $N$ elements in vector $E[\Theta(t)]$ will achieve agreement $E[\Theta_1(t)] = E[\Theta_2(t)] = \cdots = E[\Theta_N(t)]$ and the rate of convergence is only determined by the strength of the global cue $g$.

In the following, we will show the variance $V[\Theta(t)]$ is finite and will converge with a convergence rate determined by the strength of the global cue. Here our derivation is divided into two steps. In the first step, we will show the first item in (15), i.e., $e^{At}V[\Theta(0)]e^{At}$ will converge to zero with a convergence rate determined by the strength of the global cue $g$, and then in the second step, we will prove the second item in (15), i.e., \[\int_0^t e^{A(t-\tau)}\Sigma^T \Sigma e^{A(t-\tau)} d\tau\] is always finite and will converge to a constant with a convergence rate determined by the strength of the global cue $g$.

Since $\theta_1$ is noise-free, it can be easily obtained that $V[\Theta(0)]$ is of the following form
\[V[\Theta(0)] = \begin{bmatrix} 0 & 0 \\ 0 & V_{2-N}(0) \end{bmatrix} \]
with $V_{2-N}(0)$ an $(N - 1) \times (N - 1)$ dimensional matrix. Making use of the fact that
\[P^{-1} \begin{bmatrix} 0 & 0 \\ 0 & V_{2-N}(0) \end{bmatrix} P^{-T} = \begin{bmatrix} 0 & 0 \\ 0 & V_{2-N}(0) \end{bmatrix} \]
where $P$ is given in (8), we have
\[e^{At}V[\Theta(0)]e^{At} = e^{P^{-1}A^{-1}t} \begin{bmatrix} 0 & 0 \\ 0 & V_{2-N}(0) \end{bmatrix} e^{P^{-1}A^{-1}t}\]
\[= P \begin{bmatrix} e^0 & 0 \\ 0 & e^{\lambda_{2-N}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & V_{2-N}(0) \end{bmatrix} \times \begin{bmatrix} e^0 & 0 \\ 0 & e^{\lambda_{2-N}} \end{bmatrix} P^T \]
\[= P \begin{bmatrix} 0 & 0 \\ e^{\lambda_{2-N}}V_{2-N}(0)e^{\lambda_{2-N}t} \end{bmatrix} P^T. \] (17)

According to the analysis in Section 3.1, the real parts of the eigenvalues of $A_{2-N}$ are all no larger than $-g$, so we know $e^{\lambda_{2-N}}V_{2-N}(0)e^{\lambda_{2-N}t}$ is finite and will converge to zero with a convergence rate $2g$. Therefore $e^{At}V[\Theta(0)]e^{At}$ is finite and will converge to zero with a convergence rate $2g$.

Now we will prove that the second item in (15), i.e., \[\int_0^t e^{A(t-\tau)}\Sigma^T \Sigma e^{A(t-\tau)} d\tau\] is finite and will converge to a constant with a convergence rate determined by the strength of the global cue $g$.

By using (9) and the fact that $P^{-1} \Sigma \Sigma^T P = \Sigma \Sigma^T$, we have
\[e^{A(t-\tau)}\Sigma^T \Sigma e^{A(t-\tau)} = e^{P^{-1}A^{-1}(t-\tau)}\Sigma^T e^{P^{-1}A^{-1}(t-\tau)} \Sigma \Sigma^T = P e^{A(t-\tau)} \Sigma \Sigma^T P e^{A(t-\tau)} P^T. \] (18)

Denote $\Sigma = \text{diag}(0, \Sigma_{2-N})$ with
$\Sigma_{2-N} = \text{diag}(\sigma_2, \ldots, \sigma_N)$ then we have
\[P e^{A(t-\tau)} \Sigma \Sigma^T P e^{A(t-\tau)} P^T = P \begin{bmatrix} e^0 & 0 \\ 0 & e^{\lambda_{2-N}(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \Sigma_{2-N} \Sigma_{2-N} \end{bmatrix} \times \begin{bmatrix} e^0 & 0 \\ 0 & e^{\lambda_{2-N}(t-\tau)} \end{bmatrix} P^T \]
\[= P \begin{bmatrix} 0 & 0 \\ e^{\lambda_{2-N}(t-\tau)} \Sigma_{2-N} \Sigma_{2-N} e^{\lambda_{2-N}(t-\tau)} \end{bmatrix} P^T. \] (19)

Therefore it follows
\[\int_0^t e^{A(t-\tau)}\Sigma^T \Sigma e^{A(t-\tau)} d\tau = P \begin{bmatrix} 0 & 0 \\ 0 & Q_e \end{bmatrix} P^T, \]
\[Q_e = \int_0^t e^{\lambda_{2-N}(t-\tau)} \Sigma_{2-N} \Sigma_{2-N} e^{\lambda_{2-N}(t-\tau)} d\tau. \] (20)

According to the analysis in Section 3.1, all the real parts of the eigenvalues of $A_{2-N}$ are negative, so it follows that $Q_e$ is determined by (Karatzas & Shreve, 1991)
\[Q_e = Q - e^{\lambda_{2-N}t}Q e^{\lambda_{2-N}t}, \quad t > 0 \] (21)
where $Q$ is a finite constant and is given by
\[Q = \int_0^{\infty} e^{\lambda_{2-N}t} \Sigma_{2-N} \Sigma_{2-N} e^{\lambda_{2-N}t} d\tau. \]

Recall that the real parts of the eigenvalues of $A_{2-N}$ are no larger than $-g$, we have that $Q_e$ is finite and will converge to $Q$ with convergence rate $2g$.

Concluding the derivations above, we know the mathematical expectations will asymptotically achieve agreement with a convergence rate determined by $g$, and the variances will always be finite, and converge to a finite constant with a convergence rate determined by $g$ as well, which completes the proof. □
Remark 5. It is interesting to note that $Q$ is in fact the controllability Gramian of the subsystem $\theta_1, \ldots, \theta_N$.

Next, we estimate the variance upper-bound $Q$.

From Karatzas and Shreve (1991), $Q$ can be obtained by solving the following linear matrix equation
\[ \dot{A}_{2-N} + QA_T^{2-N} = -\Sigma_{2-N} \Sigma_T^{2-N}. \] (22)

If influences of the local cue are bidirectional, i.e.,
\[ a_{ij} = a_{ji}, \ i, j \in \{2, 3, \ldots, N\} \]
which yields
\[ \lambda_{\max}(\dot{A}_{2-N} + QA_T^{2-N}) = -2g \] (23)
or balanced (Olfati-Saber et al., 2007), i.e.,
\[ \sum_{2 \leq i \leq N} a_{ji} \forall 2 \leq i \leq N \]
which from the Gershgorin Circle Theorem yields
\[ \lambda_{\max}(\dot{A}_{2-N} + QA_T^{2-N}) \leq -2g \] (24)

then according to Komaroff (1990), we have
\[ Q \leq \lambda_{\max}(\dot{A}_{2-N} + QA_T^{2-N}) \frac{\lambda_{\max}(\Sigma_{2-N} \Sigma_T^{2-N})}{N - 1} \]
(25)

which means that the variances are still upper-bounded by a function of the strength of the global cue. When the local cue is neither symmetric nor balanced, suppose the maximal imbalance (Henning & Rautenbach, 2007) of the local cue is
\[ A = \max_{2 \leq i \leq N} \left| \sum_{2 \leq j \leq N, j \neq i} a_{ji} - \sum_{2 \leq j \leq N, j \neq i} a_{ij} \right| \]
(27)

then by using the Gershgorin Circle Theorem, we have
\[ \lambda_{\max}(\dot{A}_{2-N} + QA_T^{2-N}) \leq -2g + A. \] (28)

When the maximal imbalance is moderate ($A < 2g$), we can still estimate the upper-bound of the variances:
\[ \int_0^t e^{\Lambda(t-\tau)} \Sigma T e^{A(t-\tau)} d\tau \leq \frac{\lambda_{\max}(\Sigma_{2-N} \Sigma_T^{2-N})}{2g - A} PP^T. \] (29)

So in this case, the upper-bound of the variances is determined by the strength of the global cue and the imbalance of the local cue.

Remark 6. If Assumption 1 is not satisfied, i.e., $a_{21}, a_{31}, \ldots, a_{Kn}$ are not identical, then by using the Gershgorin Circle Theorem and a derivation similar to the proof of Theorem 1, we can obtain that
\[ \lambda_{\max}(\dot{A}_{2-N} + QA_T^{2-N}) \leq -2g_{\text{min}} \] (30)
where $g_{\text{min}}$ is given in (10). Accordingly we have
\[ Q \leq \frac{1}{2g_{\text{min}}} \lambda_{\max}(\Sigma_{2-N} \Sigma_T^{2-N}) H_{N-1} \] (31)

and
\[ \int_0^t e^{\Lambda(t-\tau)} \Sigma T e^{A(t-\tau)} d\tau \leq \frac{\lambda_{\max}(\Sigma_{2-N} \Sigma_T^{2-N})}{2g_{\text{min}}} PP^T \] (32)

which means that the variances are still upper-bounded by a function of the strength of the global cue.

4. Simulation results

In this section, simulation results are given to illustrate the analytical results. In the simulation, we considered an oscillator network composed of ten oscillators and one of them is the leader node (acting as the global cue). The strength of the global cue $g$ is constant and the strength of the local cue $a_{ij}$ ($i, j \in \{2, 3, \ldots, N\}$) are randomly chosen from the interval $[0, 1]$. As in most papers, we use the synchronization index to measure the degree of synchrony (Kuramoto, 1984; Pikovsky et al., 2001)
\[ SI = \left| 1 - \frac{1}{N} \sum_{i=1}^{N} e^{i\phi_i} \right|. \]

$SI \in [0, 1]$ reflects the degree of synchrony and will be 1 if the network is perfectly synchronized and 0 if the phases are randomly distributed.

We first considered the noise-free case. In order to compare influences of global and local cues on the rate of synchronization, we simulated the oscillator network under different strengths of global and local cues. Synchronization is defined to be achieved when $SI$ exceeds 0.9. In our simulations, the synchronization index $SI$ converged to 1 as shown in Fig. 1, which confirms the synchronization claim in Theorem 1. The synchronization times under different circumstances are given in Table 1. In the table, $k \times a_j$ ($k = 1, 2, \ldots, 10$) denotes that we made the strength of the local cue $a_{ij}$ ($\forall i, j \in \{2, 3, \ldots, N\}$) $k$ times larger than the original ones. So each row gives the synchronization times under a constant strength of global cue and different strengths of local cues, and meanwhile each column gives the synchronization times under different global cues and a constant local cue. When there is global cue ($g \neq 0$), from Table 1, it is clear that the rate of synchronization will increase with an increase in the strength of the global cue. However, under a constant strength of the global cue, the strongest local cue may not produce the fastest rate of synchronization, namely the shortest synchronization time, which are emphasized in bold in each row with a non-zero $g$ in Table 1. When there is no global cue ($g = 0$), the rate of synchronization will increase with an increase in the local cue, which confirms Remark 2.

The simulation results corroborate the results on the synchronization of the mammalian circadian clock in To et al. (2007): VIP input acts as the local cue, external entrainment light–dark cycles act as the global cue, and it is shown that changing the VIP input from pulsed input to constant input (making the local coupling stronger) will impair synchronization, whereas introducing light–dark cycles will enhance synchronization. Moreover, in molecular-dynamics simulations of bimolecular systems (Karplus & McCammon, 2002), it is found that the fastest rate of synchronization is achieved when the local coupling strength is moderate.
Table 1
Synchronization times under different strengths of global and local cues – Noise-free case.

<table>
<thead>
<tr>
<th>SGC</th>
<th>1 × αg</th>
<th>2 × αg</th>
<th>3 × αg</th>
<th>4 × αg</th>
<th>5 × αg</th>
<th>6 × αg</th>
<th>7 × αg</th>
<th>8 × αg</th>
<th>9 × αg</th>
<th>10 × αg</th>
</tr>
</thead>
<tbody>
<tr>
<td>g = 0</td>
<td>1.3167</td>
<td>0.6584</td>
<td>0.4390</td>
<td>0.3252</td>
<td>0.2634</td>
<td>0.2195</td>
<td>0.1882</td>
<td>0.1647</td>
<td>0.1464</td>
<td>0.1318</td>
</tr>
<tr>
<td>g = 1</td>
<td>0.4827</td>
<td>0.4671</td>
<td>0.4674</td>
<td>0.4676</td>
<td>0.4676</td>
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<td>0.4676</td>
</tr>
<tr>
<td>g = 2</td>
<td>0.2807</td>
<td>0.2414</td>
<td>0.2344</td>
<td>0.2336</td>
<td>0.2337</td>
<td>0.2337</td>
<td>0.2338</td>
<td>0.2338</td>
<td>0.2338</td>
<td>0.2338</td>
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<tr>
<td>g = 4</td>
<td>0.1719</td>
<td>0.1404</td>
<td>0.1268</td>
<td>0.1207</td>
<td>0.1182</td>
<td>0.1172</td>
<td>0.1169</td>
<td>0.1169</td>
<td>0.1169</td>
<td>0.1169</td>
</tr>
<tr>
<td>g = 8</td>
<td>0.1028</td>
<td>0.0850</td>
<td>0.0764</td>
<td>0.0703</td>
<td>0.0662</td>
<td>0.0635</td>
<td>0.0616</td>
<td>0.0604</td>
<td>0.0596</td>
<td>0.0592</td>
</tr>
<tr>
<td>g = 16</td>
<td>0.0585</td>
<td>0.0514</td>
<td>0.0466</td>
<td>0.0410</td>
<td>0.0403</td>
<td>0.0382</td>
<td>0.0365</td>
<td>0.0352</td>
<td>0.0341</td>
<td>0.0322</td>
</tr>
</tbody>
</table>

* SGC-Thickness of the global cue.

Table 2
Synchronization times under different strengths of global and local cues – Noisy case.

<table>
<thead>
<tr>
<th>SGC</th>
<th>1 × αg</th>
<th>2 × αg</th>
<th>3 × αg</th>
<th>4 × αg</th>
<th>5 × αg</th>
<th>6 × αg</th>
<th>7 × αg</th>
<th>8 × αg</th>
<th>9 × αg</th>
<th>10 × αg</th>
</tr>
</thead>
<tbody>
<tr>
<td>g = 0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>g = 1</td>
<td>0.4836</td>
<td>0.4669</td>
<td>0.4670</td>
<td>0.4670</td>
<td>0.4670</td>
<td>0.4670</td>
<td>0.4670</td>
<td>0.4670</td>
<td>0.4670</td>
<td>0.4670</td>
</tr>
<tr>
<td>g = 2</td>
<td>0.2815</td>
<td>0.2430</td>
<td>0.2359</td>
<td>0.2352</td>
<td>0.2352</td>
<td>0.2352</td>
<td>0.2354</td>
<td>0.2354</td>
<td>0.2354</td>
<td>0.2354</td>
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<tr>
<td>g = 4</td>
<td>0.1720</td>
<td>0.1403</td>
<td>0.1267</td>
<td>0.1206</td>
<td>0.1179</td>
<td>0.1170</td>
<td>0.1167</td>
<td>0.1166</td>
<td>0.1166</td>
<td>0.1166</td>
</tr>
<tr>
<td>g = 8</td>
<td>0.1028</td>
<td>0.0850</td>
<td>0.0764</td>
<td>0.0702</td>
<td>0.0661</td>
<td>0.0634</td>
<td>0.0617</td>
<td>0.0605</td>
<td>0.0598</td>
<td>0.0593</td>
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<tr>
<td>g = 16</td>
<td>0.0585</td>
<td>0.0515</td>
<td>0.0467</td>
<td>0.0431</td>
<td>0.0404</td>
<td>0.0383</td>
<td>0.0365</td>
<td>0.0352</td>
<td>0.0341</td>
<td>0.0322</td>
</tr>
</tbody>
</table>

* SGC-Thickness of the global cue.

It is worthwhile to mention that when the local coupling strength is strong, the synchronization rate in the global-cue-free case is even faster than the case with a weak global cue (the first and second entries of the last column in Table 1). This shows that when there are both global and local cues, the global cues dominate the synchronization process. Moreover, when there is no global cue, it has been established in some cases that local coupling plays an important role in ensuring robust synchronization with respect to model uncertainties. But under the formulation in the paper, our simulation results demonstrate that in the presence of Gaussian white noise in the natural frequencies, the global cue is much more important than the local cue in ensuring synchronization of the whole system. (As shown in the first row of Table 2, synchronization is lost when there are only local cues but no global cue.)

5. Conclusions

The influence of global and local cues on the rate of synchronization is studied. This has important ramifications for the study of biological oscillators. It is proved through analysis and numerical simulations that only the strength of the global cue, but not the strengths of global and local cues together, determines the rate of synchronization. More specifically, a stronger global cue means a faster rate of synchronization but a stronger local cue does not necessarily bring a faster rate of synchronization. This points out a way to elucidate complex synchronization properties in connected oscillator networks, such as biological oscillator networks. The results apply both to the noise-free case and to the case with Gaussian white noise in oscillators’ natural frequencies.

References


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